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HARMONIC ANALYSIS OF SPACETIME WITH HYPERCHARGE AND ISOSPIN

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Abstract

In analogy to the harmonic analysis for the Poincaré group with its irreducible representations characterizing free particles, the harmonic analysis for a nonlinear spacetime model as homogeneous space of the extended Lorentz group $\mathbf{GL}(\mathbb{C}^2)$ is given. What the Dirac energy-momentum measures are for free particles, are multipole measures in the analysis of nonlinear spacetime - they are related to spacetime interactions. The representations induced from the nonlinear spacetime fixgroup $\mathbf{U}(2)$ connect representations of external (spacetime-like) degrees of freedom with those of internal (hypercharge-isospinlike) ones as seen in the standard model of electroweak and strong interactions. The methods used are introduced and exemplified with the nonrelativistic Kepler dynamics in an interpretation as harmonic analysis of time and position functions.

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Contents

Chapter 1

Orbits of Time and Space

This chapter serves as a - very long - introduction. It treats with the nonrelativistic Kepler dynamics - classical and quantum. The Kepler dynamics with the $\frac{1}{r}$-potential has, on the one side, enough structure and, on the other side, is familiar enough to practice the language which will be talked in the following chapters for particles and interactions, related to relativistic spacetime. It exemplifies the necessary concepts - operational symmetries and orbits, realizations and representations[4, 9, 11, 13] of, especially, time and position, compact and noncompact for spherical, flat and hyperbolic spaces, scattering and bound states, special functions as matrix elements[19], homogeneous spaces, invariant measures, multipole measures, harmonic expansions etc.

1.1 Symmetries of the Kepler Dynamics

The Kepler Hamiltonian

$$H = \frac{\vec{p}^2}{2} + \frac{\delta}{r}, \quad \delta = \pm 1 \quad \text{(repulsion, attraction)}$$

has, in addition to rotation invariance with conserved angular momentum $\vec{\mathcal{L}}$, the Lenz-Runge invariance with conserved Lenz-Runge vector $\vec{\mathcal{P}}$ (for the sun system called perihelion vector and perihelion conservation - no rosettes) given with the Poisson Lie-bracket $[f,g]_P = \frac{\partial f}{\partial \vec{p}}\frac{\partial g}{\partial \vec{x}} - \frac{\partial f}{\partial \vec{p}}\frac{\partial f}{\partial \vec{x}}$ from $[p^a, x^b]_P = \delta^{ab}$

$$\vec{\mathcal{L}} = \vec{x} \times \vec{p}, \quad \vec{\mathcal{P}} = \vec{p} \times \vec{\mathcal{L}} + \delta \frac{\vec{x}}{r} \Rightarrow \begin{cases} [H, \vec{\mathcal{P}}]_P &= 0 \\ [\mathcal{L}^a, \mathcal{P}^b]_P &= -\epsilon^{abc}\mathcal{P}^c \\ [\mathcal{P}^a, \mathcal{P}^b]_P &= 2H\epsilon^{abc}\mathcal{L}^c \end{cases}$$

The Kepler Hamiltonian should be compared in the following with the free Hamiltonian (constant potential)

$$H_0 = \frac{\vec{p}^2}{2} + V_0 \Rightarrow \begin{cases} [H_0, \vec{p}]_P &= 0 \\ [\mathcal{L}^a, p^b]_P &= -\epsilon^{abc}p^c \\ [p^a, p^b]_P &= 0 \end{cases}$$

As seen below, the Lenz-Runge vector $\vec{\mathcal{P}}$ expands the translations $\vec{p}$.

With the Hamiltonian in the Lenz-Runge vector brackets three different types for the energy values from the spectrum of the Kepler Hamiltonian have to be distinguished to classify its possible symmetries. In all three cases $E = 0$, $E > 0$ and $E < 0$, the dynamics is characterized by a real 6-dimensional invariance Lie algebra of rank 2, i.e. with two independent invariants.

For trivial energy, the symmetry is - as for the free Hamiltonian - the semidirect Euclidean structure with the rotations and translations in three dimensions[1]

$$\mathrm{spec}\, H \ni E = 0: \quad \begin{cases} [l^a, l^b] = -\epsilon^{abc} l^c, \ [l^a, p^b] = -\epsilon^{abc} p^c, \ [p^a, p^b] = 0 \\ \text{invariants: } \vec{p}^2, \quad \vec{l}\vec{p} \\ \text{representation: } l^a \longmapsto \mathcal{L}^a, \quad p^a \longmapsto \mathcal{P}^a \end{cases}$$
$$\text{Lie algebra:} \quad A_1^c \ \vec{\oplus} \ \mathbb{R}^3 = \log[\mathbf{SO}(3) \ \vec{\times}\, \mathbb{R}^3] \cong \mathbb{R}^6$$

The angular momentum invariant $\vec{l}^2$ is no translation invariant.

A nontrivial energy can be used to renormalize the Lenz-Runge vector

$$\begin{array}{c} \mathrm{spec}\, H \ni E \neq 0, \quad \vec{\mathcal{B}} = \frac{\vec{\mathcal{P}}}{\sqrt{2|H|}}: \\ \epsilon(E) = \frac{E}{|E|} = \frac{H}{|H|} \end{array} \quad \begin{cases} [\mathcal{L}^a, \mathcal{L}^b]_P &= -\epsilon^{abc} \mathcal{L}^c \\ [\mathcal{L}^a, \mathcal{B}^b]_P &= -\epsilon^{abc} \mathcal{B}^c \\ [\mathcal{B}^a, \mathcal{B}^b]_P &= \epsilon(E) \epsilon^{abc} \mathcal{L}^c \end{cases}$$

Positive energies lead to scattering orbits (classical and quantal), there arises the Lie algebra of the noncompact Lorentz group with the Lenz-Runge vector defining the 'boosts' (called so in analogy - no special relativistic transformations)

$$E > 0: \quad \begin{cases} [l^a, l^b] = -\epsilon^{abc} l^c = -[b^a, b^b], \quad [l^a, b^b] = -\epsilon^{abc} b^c \\ \text{invariants: } \vec{l}^2 - \vec{b}^2, \ \vec{l}\vec{b} \\ \text{representation: } l^a \longmapsto \mathcal{L}^a, \quad b^a \longmapsto \mathcal{B}^a \end{cases}$$
$$\text{Lie algebra:} \quad A_1^c \ \oplus \ i A_1^c \cong \log \mathbf{SO}(1,3) \cong \mathbb{R}^6$$

For negative energies, leading to bound orbits (classical and quantal), the symmetries constitute the compact Lie algebra of $\mathbf{SO}(4)$, locally isomorphic to $\mathbf{SO}(3) \times \mathbf{SO}(3)$

$$E < 0: \quad \begin{cases} [l^a, l^b] = -\epsilon^{abc} l^c = [m^a, m^b], \quad [l^a, m^b] = -\epsilon^{abc} m^c \\ \vec{l}_\pm = \frac{\vec{l} \pm \vec{m}}{2} \Rightarrow [l^a_\pm, l^b_\pm] = -\epsilon^{abc} l^c_\pm, \ [l^a_+, l^b_-] = 0 \\ \text{invariants: } \vec{l}^2 + \vec{m}^2 = 2(\vec{l}_+^2 + \vec{l}_-^2), \ \vec{l}\vec{m} = \vec{l}_+^2 - \vec{l}_-^2 \\ \text{representation: } l^a \longmapsto \mathcal{L}^a, \quad m^a \longmapsto \mathcal{B}^a \end{cases}$$
$$\text{Lie algebra:} \quad A_1^c \ \oplus \ A_1^c \cong \log[\mathbf{SO}(3) \times \mathbf{SO}(3)] \cong \mathbb{R}^6$$

The transition from positive to negative energies can be formulated as a transition from real momenta to imaginary 'momenta'

$$\sqrt{2E} = \begin{cases} |\vec{q}|, & E > 0, \quad \text{scattering orbits, real momenta} \\ i|Q|, & E < 0, \quad \text{bound orbits, imaginary 'momenta'} \end{cases}$$

[1] $\log G$ denotes the Lie algebra of a Lie group G, e.g. $\log \mathbf{SU}(2) = A_1^c$. $G \ \vec{\times}\, H$ denotes a semidirect product with the group G acting upon the group H.

The three Kepler Lie groups can be decomposed - as manifolds - with rotation group classes[2]

$$\begin{aligned}\mathbf{SO}(4) &\cong \mathbf{SO}(3) \times \mathbf{SO}(4)/\mathbf{SO}(3)\\ \mathbf{SO}_0(1,3) &\cong \mathbf{SO}(3) \times \mathbf{SO}_0(1,3)/\mathbf{SO}(3)\end{aligned}$$

The rotation group itself is a product of axial rotations $\mathbf{SO}(2)$ with the 2-sphere[3] $\Omega^2 \cong \mathbf{SO}(3)/\mathbf{SO}(2)$ for the angular momentum direction, i.e. Ω^2 is the orientation manifold of the axial rotations. The Lenz-Runge operations, compact as 3-sphere $\Omega^3 = \mathbf{SO}(4)/\mathbf{SO}(3)$ and noncompact as 3-hyperboloid $\mathcal{Y}^3 = \mathbf{SO}_0(1,3)/\mathbf{SO}(3)$, are the orientation manifolds - or equivalence classes - of the rotation group. Also these symmetric spaces have a manifold decomposition into a characteristic Abelian subgroup and a 2-sphere for the Lenz-Runge vector direction

$$\begin{pmatrix}\text{spherical}\\ \text{flat, Euclidean}\\ \text{hyperbolic}\end{pmatrix}: \quad \begin{pmatrix}\mathbf{SO}(4)\\ \mathbf{SO}(3) \vec{\times} \mathbb{R}^3\\ \mathbf{SO}_0(1,3)\end{pmatrix} \cong \mathbf{SO}(2) \times \Omega^2 \times \begin{pmatrix}\mathbf{SO}(2)\\ \mathbb{R}\\ \mathbf{SO}_0(1,1)\end{pmatrix} \times \Omega^2$$

The 2-dimensional Abelian subgroups reflect the rank 2 with the two independent invariants. The rotation group $\mathbf{SO}(3)$ in all groups determines the angular momentum. The representation of the 2nd Abelian factor decides on the spherical, parabolic (or flat) and hyperbolic orbits.

1.2 Classical Time Orbits

In classical theories one is primarily interested in the time orbits[4] the mass points perform in position space $\mathbb{R} \ni t \longmapsto \vec{x}(t) \in \mathbb{R}^3$, mathematically in the irreducible realizations of the time translation group $\mathbb{R}$. The characterizing eigenvalues, i.e. the invariant energies, are classically imposed by boundary or initial conditions.

1.2.1 Time Orbits in Position Space

The functions of position and momentum $(\vec{x}, \vec{p})$ build an associative unital algebra, for a classical framework commutative with the pointwise product. The problems with the Kepler potential $\frac{1}{r}$ at the origin $\vec{x} = 0$ are neglected, they deserve a more careful discussion. This algebra has a noncommutative Lie algebra structure with the Poisson bracket. Therewith the three Kepler Lie algebras $\log \mathbf{SO}(4)$, $\log \mathbf{SO}_0(1,3)$ and $\log[\mathbf{SO}(3) \vec{\times} \mathbb{R}^3]$ act adjointly on this algebra.

[2] The subindex in $\mathbf{SO}_0(1,s)$ denotes the unit connection component.

[3] The s-sphere and the s-hyperboloid are denoted by $\Omega^s \cong \mathbf{SO}(1+s)/\mathbf{SO}(s)$ and $\mathcal{Y}^s \cong \mathbf{SO}_0(1,s)/\mathbf{SO}(s)$ resp. for $s = 1, 2, \dots$ with $\mathcal{Y}^{1+s} \cong \mathcal{Y}^1 \times \Omega^s$.

[4] A group G, realized by bijections (permutations, automorphisms) on a set S, defines an orbit $G \bullet x \subseteq S$ for each element $x \in S$ and a fixgroup $G_x \subseteq G$ with $G \bullet x \cong G/G_x$. The group action decomposes the set into disjoint orbits $G \bullet x \in S/G$. For a vector space $S = V$, the orbit displays matrix elements $\langle \omega, g \bullet x \rangle$ of the group representation $G \ni g \longmapsto \mathbf{GL}(V)$.

The products of angular momentum with position, momentum and Lenz-Runge vector vanish (orthogonality)

$$\vec{\mathcal{L}}\vec{x} = 0, \quad \vec{\mathcal{L}}\vec{p} = 0, \quad \vec{\mathcal{L}}\vec{\mathcal{P}} = 0$$

i.e., position $\vec{x}$, momentum $\vec{p}$ and Lenz-Runge vector $\vec{\mathcal{P}}$ are in the position orbit plane. For gravity in the sun system the orbit planarity following from rotation invariance constitutes Kepler's 1st law.

The invariant squares of angular momentum and perihelion vector combine the Hamiltonian and determine the energy E by angular momentum value L and perihelion value P

$$\begin{aligned} \vec{\mathcal{P}}^2 &= 1 + 2H\vec{\mathcal{L}}^2 \Rightarrow E = \tfrac{P^2-1}{2L^2} \text{ with } |\vec{\mathcal{P}}| = P,\ |\vec{\mathcal{L}}| = L \\ -\tfrac{1}{2H} &= \vec{\mathcal{L}}^2 - \epsilon(E)\vec{\mathcal{B}}^2 \end{aligned}$$

The time orbits in position space are conic sections, described by polar equations with one focus as origin (2nd Kepler law)

$$\begin{aligned} \vec{\mathcal{P}}\vec{x} = Pr\cos\varphi &= (\vec{p}\times\vec{\mathcal{L}})\vec{x} + \delta r = L^2 + \delta r \\ \Rightarrow r(\varphi) &= \tfrac{L^2}{P\cos\varphi - \delta} \text{ with } \delta = \pm 1 \end{aligned}$$

$\vec{x}$ shows to the peri- and aphelion for $\varphi = 0$ and $\varphi = \pi$ resp. The invariants can be expressed by perihelion distance r_0 and momentum p_0 as possible initial conditions

$$\text{for } \varphi = 0: \quad \vec{x}_0\vec{p}_0 = 0 \Rightarrow \begin{cases} L &= r_0 p_0 \\ P &= r_0 p_0^2 + \delta \\ E &= \frac{p_0^2}{2} + \frac{\delta}{r_0} \end{cases}$$

The connection between Cartesian $\vec{x} = (x, y, 0)$ and and polar equations is given in the following table

energy $E = \frac{P^2-1}{2L^2}$	$E < 0$	$E > 0$	$E = 0$
group orbit	$\mathbf{SO}(2)$, ellipse	$\mathbb{I}(2) \times \mathbf{SO}_0(1,1)$, hyperbola	$\mathbb{R}$, parabola
Cartesian equation	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $a \geq b$	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ (right and left branch)	$y^2 = -2dx$ $d > 0$
foci distance $2c$	$c^2 = a^2 - b^2$	$c^2 = a^2 + b^2$	(one focus)
polar equation (with pole in the right focus)	$r(\varphi) = \frac{L^2}{P\cos\varphi+1}$ $0 < P < 1$	$r_{R,L}(\varphi) = \frac{L^2}{P\cos\varphi\pm1}$ $P > 1$	$r(\varphi) = \frac{L^2}{\cos\varphi+1}$ $P = 1$
distance of pole to peri- and aphelion	$(a-c, a+c)$ $= (r(0), r(\pi))$ $= (\frac{L^2}{1+P}, \frac{L^2}{1-P})$	$(c-a, c+a)$ $= (r_R(0), r_L(0))$ $= (\frac{L^2}{P+1}, \frac{L^2}{P-1})$	$\frac{d}{2}$ $= r(0)$ $= \frac{L^2}{2}$
$(a, c, b) =$	$(-\frac{1}{2E}, -\frac{P}{2E}, \frac{L}{\sqrt{-2E}})$	$(\frac{1}{2E}, \frac{P}{2E}, \frac{L}{\sqrt{2E}})$	
$(L^2, P) =$	$(\frac{b^2}{a}, \frac{c}{a})$	$(\frac{b^2}{a}, \frac{c}{a})$	$(d, 1)$

Cartesian and Polar Equations for Conic Sections

The squared length of the perihelion vector is the discriminant of the 2nd order polynomial which occurs in the equation of motion

$$\begin{aligned} \tfrac{dr}{dt} = [H, r]_P = p_r, \quad p_r^2 &= \tfrac{2Er^2 - 2\delta r - L^2}{r^2} \\ -\det\begin{pmatrix} 2E & \delta \\ \delta & -L^2 \end{pmatrix} &= 1 + 2EL^2 = P^2 \text{ for } \delta^2 = 1 \end{aligned}$$

Attraction $\delta = -1$, e.g. in gravity or with charge numbers of opposite sign $z_1 z_2 < 0$ in electrostatics, can come with negative and positive energies for compact and noncompact orbits resp.

$$\delta = -1: \;\; P^2 = 1 + 2EL^2 \begin{cases} < 1 \iff & -\frac{1}{2L^2} \le E < 0 \\ & \text{(ellipse)} \\ = 1 \iff & E = 0 \\ & \text{(parabola)} \\ > 1 \iff & E > 0 \\ & \text{(hyperbola branch around pole)} \end{cases}$$

Repulsion $\delta = 1$, e.g. for charge numbers of equal sign $z_1 z_2 > 0$, has positive energies only (noncompact orbits)

$$\delta = 1: \;\; P^2 = 1 + 2EL^2 > 1 \iff \begin{array}{l} E > 0 \\ \text{(hyperbola branch, not around pole)} \end{array}$$

Since the angular momentum is twice the time change of the orbit area

$$\vec{\mathcal{L}} = \vec{x} \times d_t \vec{x} = 2 d_t \vec{\mathcal{A}}$$

the orbit area $A = |\vec{\mathcal{A}}|$ and the orbit time T for ellipses are related to the conserved angular momentum which relates the large axis of the ellipse to the orbit time (Kepler's 3rd law)

$$L = 2\frac{A}{T} = 2\frac{\pi ab}{T} \Rightarrow a^3 = (\frac{T}{2\pi})^2 = -\frac{1}{8E^3}$$

For the free theory the orbits are lines

$$\delta = 0: \quad E = \frac{\vec{p}^2}{2}, \;\; r(\varphi) = \frac{r(0)}{\cos\varphi} \;\; \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \xi & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

1.2.2 Orbits as Time Classes

A dynamics leads to realizations of the time translation group. The solutions in classical physics are irreducible time orbits in position space. They have to be isomorphic to quotient groups of time $\mathbb{R}$ with the kernel K of the time realization

$$\mathbb{R} \ni t \longmapsto \vec{x}(t) \in \mathbb{R}^3, \quad \vec{x}[\mathbb{R}] \cong \mathbb{R}/K \subset \mathbb{R}^3$$

The $\mathbb{R}$-subgroups which can arise as kernels for Lie quotient groups are the full group $\mathbb{R}$, the trivial group $\{0\}$ and - up to isomorphy - the discrete group $\mathbb{Z}$.

All possible quotient groups $\mathbb{R}/K$ (time equivalence classes) for time realizations are seen in the sky as solutions of the Kepler dynamics. The trivial time representation is seen in our sun, assumed infinitely heavy, with trivial

orbit $\mathbb{R}/\mathbb{R} \cong \{1\}$ and energy $E = -\infty$. The compact unfaithful time realizations with kernel $\mathbb{Z}$ give bound orbits (planets on ellipses), isomorphic to the torus $\mathbf{U}(1)$. For the earth, $\mathbb{Z}$ counts the years. The noncompact faithful realizations, isomorphic to full $\mathbb{R}$, give scattering orbits (never returning comets on one branch of an hyperbola or on a parabola).

The conic sections (ellipses and hyperbolas) for the planar orbits have the metrical tensors

$$\frac{x^2}{a^2} \pm \frac{y^2}{b^2} = 1, \ \begin{pmatrix} \frac{1}{a^2} & 0 \\ 0 & \pm\frac{1}{b^2} \end{pmatrix} = \begin{pmatrix} 4E^2 & 0 \\ 0 & -\frac{2E}{L^2} \end{pmatrix} = \frac{4E^2}{L\sqrt{2|E|}} \begin{pmatrix} \eta & 0 \\ 0 & \pm\frac{1}{\eta} \end{pmatrix}$$

with the ratio of the units for the two directions $\eta = \frac{b}{a} = L\sqrt{2|E|}$ the product of angular momentum with energy. The orbits in position space can be parametrized as follows

$$\begin{array}{lll} \text{ellipses:} & \mathbf{SO}(2) \cong \mathbb{R}/\mathbb{Z}: & \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\chi & -\eta\sin\chi \\ \frac{1}{\eta}\sin\chi & \cos\chi \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \text{hyperbolas:} & \mathbf{SO}_0(1,1) \cong \mathbb{R}: & \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cosh\psi & \eta\sinh\psi \\ \frac{1}{\eta}\sinh\psi & \cosh\psi \end{pmatrix} \begin{pmatrix} \pm 1 \\ 0 \end{pmatrix} \end{array}$$

The time parametrization comes via time dependent group parameters, $t \longmapsto \chi(t), \psi(t)$.

How are the time orbits, i.e. the quotient groups $\mathbb{R}/K$, related to the full Kepler groups $\mathbf{SO}(4)$, $\mathbf{SO}_0(1,3)$ and $\mathbf{SO}(3) \,\vec{\times}\, \mathbb{R}^3$ acting upon the algebra with the position-momentum functions? An individual solution of a dynamics (time realizations), here one orbit $\vec{x}$ in position space, has not to have all the invariances G of the Hamiltonian. $\vec{x}$-equivalent solutions are on the orbit $G \bullet \vec{x} \cong G/H$ of the invariance group G in the solution space (not in position space) with a remaining subgroup H-symmetry. E.g., by rotating the initial conditions of one solution for a rotation invariant Hamiltonian one obtains an equivalent solution. The fixgroup H of a solution is the G-subgroup which leaves the orbit in position space invariant - as a whole, not its individual points. The 6-parametric invariance of the Kepler dynamics is - up to the trivial solution with position and momentum $(\vec{x}, \vec{p}) = (0,0)$ - broken to a 1-parametric fixgroup symmetry since the choice of angular momentum and Lenz-Runge vectors $(\vec{\mathcal{L}}, \vec{\mathcal{P}})$ (six parameters) with $\vec{\mathcal{L}}\vec{\mathcal{P}} = 0$ (one condition) to determine an orbit comes from a 5-parametric manifold

$$\begin{array}{rlll} H & = \mathbf{SO}(2), & \mathbf{SO}_0(1,1), & \mathbb{R} \\ G/H & = \mathbf{SO}(4)/\mathbf{SO}(2), & \mathbf{SO}_0(1,3)/\mathbf{SO}_0(1,1), & [\mathbf{SO}(3) \,\vec{\times}\, \mathbb{R}^3]/\mathbb{R} \end{array}$$

The nontrivial orbits $t \longmapsto \vec{x}(t)$ in position as quotient groups $\mathbb{R}/K$ of time are isomorphic to the characterizing fixgroups H in the Kepler groups. In all three cases, the solution degeneracy is $G/H \cong \mathbf{SO}(3) \times \Omega^2$ with all three compact rotation parameters for angular momentum $\vec{\mathcal{L}}$ and two compact parameters for the direction of the Lenz-Runge vector $\vec{\mathcal{P}}$.

1.2.3 Two Sided Contraction to the Free Theory

The complex 6-dimensional group $\mathbf{SO}(\mathbb{C}^4)$, e.g. in the defining 4-dimensional representation for its Lie algebra

$$l(\vec{\varphi},\vec{\psi}) = \vec{\varphi}\vec{L} + \vec{\psi}\vec{B} = \left(\begin{array}{c|ccc} 0 & \psi_1 & \psi_2 & \psi_3 \\ \hline \psi_1 & 0 & \varphi_3 & -\varphi_2 \\ \psi_2 & -\varphi_3 & 0 & \varphi_1 \\ \psi_3 & \varphi_2 & -\varphi_1 & 0 \end{array}\right) \in \log \mathbf{SO}(\mathbb{C}^4)$$

with complex rank 2 and the corresponding two invariant bilinear forms $\kappa_{1,2}$ from the coefficients of the characteristic polynomial

$$\begin{aligned} \det [l(\vec{\varphi},\vec{\psi}) - \lambda \mathbf{1}_4] &= \lambda^4 + \lambda^2(\vec{\varphi}^2 - \vec{\psi}^2) - (\vec{\varphi}\vec{\psi})^2 \\ \Rightarrow & \left\{ \begin{array}{ll} \kappa_1(l,l) & = -\operatorname{tr} l \circ l = \vec{\varphi}^2 - \vec{\psi}^2 \\ \kappa_2(l,l) & = \sqrt{-\det l} = \vec{\varphi}\vec{\psi} \end{array} \right. \end{aligned}$$

has as real forms the groups

$$\begin{array}{ll} \mathbf{SO}_0(1,3) & \text{with } \varphi_a \in \mathbb{R},\ \psi_a \in \mathbb{R} \\ \mathbf{SO}(4) & \text{with } \varphi_a \in \mathbb{R},\ \psi_a = i\chi_a \in i\mathbb{R} \end{array}$$

Both groups are expansions of the non-semisimple Euclidean group $\mathbf{SO}(3) \,\vec{\times}\, \mathbb{R}^3$

$$\left(\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \hline \xi_1 & 0 & \varphi_3 & -\varphi_2 \\ \xi_2 & -\varphi_3 & 0 & \varphi_1 \\ \xi_3 & \varphi_2 & -\varphi_1 & 0 \end{array}\right) \in \log[\mathbf{SO}(3) \,\vec{\times}\, \mathbb{R}^3]$$

with the 'boosts' $\vec{\psi}$ or the additional 'internal rotations' $\vec{\chi}$ as 'unflattened, expanded translations'.

Vice versa, the translations $\mathbb{R}^3$ arise by Inönü-Wigner contraction[23] to the Galilei group as tangent space both of the compact 3-sphere $\Omega^3 \cong \mathbf{SO}(4)/\mathbf{SO}(3) \cong \mathbf{SO}(2) \times \Omega^2$, related to bound structures, and of the noncompact hyberboloid $\mathcal{Y}^3 \cong \mathbf{SO}_0(1,3)/\mathbf{SO}(3) \cong \mathbf{SO}_0(1,1) \times \Omega^2$, related to scattering structures, to the free theory with non-semisimple symmetry

$$\begin{array}{ccccc} \text{spherical} & & \text{flat} & & \text{hyperbolic} \\ \begin{pmatrix} \mathbf{SO}(4) \\ \cup \\ \mathbf{SO}(4)/\mathbf{SO}(3) \\ \cup \\ \mathbf{SO}(2) \end{pmatrix} & \xrightarrow{\eta \to 0} & \begin{pmatrix} \mathbf{SO}(3) \,\vec{\times}\, \mathbb{R}^3 \\ \cup \\ \mathbb{R}^3 \\ \cup \\ \mathbb{R} \end{pmatrix} & \xleftarrow{\eta \to 0} & \begin{pmatrix} \mathbf{SO}_0(1,3) \\ \cup \\ \mathbf{SO}_0(1,3)/\mathbf{SO}(3) \\ \cup \\ \mathbf{SO}_0(1,1) \end{pmatrix} \end{array}$$

The contraction procedure will be given explicitly for the decisive abelian subgroups (last line): The relevant contraction parameter $\eta^2 = \frac{b^2}{a^2} = 2|E|L^2$ with energy and angular momentum is the ratio of the units for the two directions in the conic sections. It is the analogue to the ratio of a time unit to a position unit $\frac{1}{c^2} = \frac{\tau^2}{\ell^2}$ in the archetypical Wigner-Inönü contraction. For the Coulomb interaction $V(r) = \frac{1}{4\pi\epsilon_0}\frac{z_1 z_2 e^2}{r}$ with $\delta = \epsilon(z_1 z_2) = \pm 1$ and the unit for the product of energy and angular momentum $[EL^2] = \frac{me^2}{\epsilon_0}$ the contraction limit is realizable by $\frac{e^2}{\epsilon_0} \to 0$. For the planetary system with gravitational interaction $-\frac{G_N M m}{r}$ and $[EL^2] = m(G_N M m)^2$ the contraction limit is realized by $G_N \to 0$.

The ratio of the units is used for a renormalization of the Lie parameters $(\chi,\psi)\stackrel{\eta}{\longmapsto}\xi$ - the analogue for the reparametrization from rapidity to velocity $\tanh\psi=\frac{v}{c}$. The contraction of the length ratio $\eta=\frac{b}{a}=L\sqrt{2|E|}\to 0$ is the analogue to the contraction to an infinite velocity $\frac{1}{c}=\frac{\tau}{\ell}\to 0$

$$\left.\begin{array}{r}E<0\\ \tan\chi=\eta\xi\end{array}\right\}: \quad \mathbf{SO}(2)\ni\begin{pmatrix}\cos\chi & \eta\, i\sin\chi\\ \frac{1}{\eta}\, i\sin\chi & \cos\chi\end{pmatrix}$$
$$=\frac{1}{\sqrt{1+\eta^2\xi^2}}\begin{pmatrix}1 & \eta^2 i\xi\\ i\xi & 1\end{pmatrix}\to\begin{pmatrix}1 & 0\\ i\xi & 1\end{pmatrix}\text{ for }\eta\to 0$$
$$\left.\begin{array}{r}E>0\\ \tanh\psi=\eta\xi\end{array}\right\}: \mathbf{SO}_0(1,1)\ni\begin{pmatrix}\cosh\psi & \eta\,\sinh\psi\\ \frac{1}{\eta}\,\sinh\psi & \cosh\psi\end{pmatrix}$$
$$=\frac{1}{\sqrt{1-\eta^2\xi^2}}\begin{pmatrix}1 & \eta^2\xi\\ \xi & 1\end{pmatrix}\to\begin{pmatrix}1 & 0\\ \xi & 1\end{pmatrix}\text{ for }\eta\to 0$$

The contracted additive group $\mathbb{R}$ comes in the multiplicative representation with nilpotent operations, typical for the non-semisimplicity of semi-direct groups

$$\mathbb{R}\ni\xi\longmapsto\begin{pmatrix}1 & 0\\ \xi & 1\end{pmatrix}=\exp\begin{pmatrix}0 & 0\\ \xi & 0\end{pmatrix}\text{ with }\begin{pmatrix}0 & 0\\ \xi & 0\end{pmatrix}^2=0$$

The contraction limit describes the line orbits of the free theory, not the parabolas of the Kepler potential.

1.3 Quantum Kepler Bound States

In the quantum case, the Kepler Hamiltonian, angular momentum and Lenz-Runge vector

$$H=\frac{\vec{p}^2}{2}+\frac{\delta}{r},\quad \vec{\mathcal{L}}=\vec{x}\times\vec{p},\quad \vec{\mathcal{P}}=\frac{\vec{p}\times\vec{\mathcal{L}}-\vec{\mathcal{L}}\times\vec{p}}{2}+\delta\frac{\vec{x}}{r}$$

give the same three Lie algebra structures as in the classical case

$$[H,\vec{\mathcal{L}}]=0,\quad [H,\vec{\mathcal{P}}]=0,\quad \left\{\begin{array}{ll}[i\mathcal{L}^a,i\mathcal{L}^b] & =-\epsilon^{abc}i\mathcal{L}^c\\ [i\mathcal{L}^a,i\mathcal{P}^b] & =-\epsilon^{abc}i\mathcal{P}^c\\ [i\mathcal{P}^a,i\mathcal{P}^b] & =2H\epsilon^{abc}i\mathcal{P}^c\end{array}\right.$$

The additional i-factor is related to the different dual normalization in the Poisson bracket $[p,x]_P=1$ and the quantum commutator $i[p,x]=1$.

Again, the squares of the angular momentum and the Lenz-Runge vector determine the Hamiltonian

$$\vec{\mathcal{P}}^2=1+2H(\vec{\mathcal{L}}^2+1)\Rightarrow -\frac{1}{2H}=1+\mathcal{L}^2-\frac{\mathcal{P}^2}{2H}$$

with an additional constant compared to the classical case.

For $\delta=-1$ and negative energies one has representations of the compact symmetry Lie algebra

$$\text{spec}\,H\ni E<0:\left\{\begin{array}{l}\vec{\mathcal{B}}=\frac{\vec{\mathcal{P}}}{\sqrt{-2H}},\quad \vec{\mathcal{J}}_\pm=\frac{\vec{\mathcal{L}}\pm\vec{\mathcal{B}}}{2}\\ [i\mathcal{J}^a_\pm,i\mathcal{J}^b_\pm]=-\epsilon^{abc}i\mathcal{J}^c_\pm,\quad [\mathcal{J}^a_+,\mathcal{J}^b_-]=0\\ \text{invariants: }\vec{\mathcal{L}}^2+\vec{\mathcal{B}}^2=2(\vec{\mathcal{J}}_+^2+\vec{\mathcal{J}}_-^2),\quad \vec{\mathcal{L}}\vec{\mathcal{B}}=\vec{\mathcal{J}}_+^2-\vec{\mathcal{J}}_-^2\end{array}\right.$$
$$\log[\mathbf{SO}(3)\times\mathbf{SO}(3)]\cong A_1^c\ \oplus\ A_1^c$$

The weight diagrams of the irreducible $\mathbf{SU}(2) \times \mathbf{SU}(2)$-representations

$$\begin{aligned}\mathbf{irrep}\,[\mathbf{SU}(2) \times \mathbf{SU}(2)] &= \mathbf{irrep}\,\mathbf{SU}(2) \times \mathbf{irrep}\,\mathbf{SU}(2)\\ &= \{(2J_1, 2J_2) \mid J_{1,2} = 0, \tfrac{1}{2}, 1, \ldots\}\end{aligned}$$

occupy $(1+2J_1)(1+2J_2)$ points of a rectangular grid.

The two invariants determine the occurring representations. The triviality of the invariant $\vec{\mathcal{L}}\vec{\mathcal{P}} = 0$ (classical orthogonality of angular momentum and Lenz-Runge vector) 'synchronizes' the centers $\mathbb{I}_2 = \{\pm 1\}$ of both $\mathbf{SU}(2)$'s (central correlation - two cycles give one bicycle) and leads to the relevant group $\mathbf{SO}(4)$

$$\frac{\mathbf{SU}(2)\times\mathbf{SU}(2)}{\mathbf{I}(2)} \cong \mathbf{SO}(4) \quad \text{with } \mathbb{I}(2) = \{(1,1), (-1,-1)\} \subset \mathbf{SU}(2) \times \mathbf{SU}(2)$$

It enforces even the equality of both $\mathbf{SU}(2)$-invariants $J_+ = J_- = J$

$$0 = \vec{\mathcal{L}}\vec{\mathcal{B}} = \vec{\mathcal{J}}_+^2 - \vec{\mathcal{J}}_-^2 \Rightarrow \langle \vec{\mathcal{J}}_+^2 \rangle = \langle \vec{\mathcal{J}}_-^2 \rangle = J(1+J), \quad J = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, \ldots$$

Therefore the energy degenerated representations are of the type $(2J, 2J)$, the multiplets of both A_1^c-representations have equal dimension $1+2J$. The $\mathbf{SU}(2)$-multiplet dimension is the principal quantum number $k = 1+2J$. The weight diagrams occupy $(1+2J)^2$ points of a square grid

$$\mathbf{irrep}\,\mathbf{SO}(4)\Big|_{\text{Kepler}} = \{(2J, 2J) \mid J = 0, \tfrac{1}{2}, 1, \ldots\}, \quad (2J, 2J) = \bigvee^{2J}(1,1)$$

The Kepler representations are the totally symmetrized products of the defining $\mathbf{SO}(4)$-representation $(1,1)$.

The energy eigenvalues are given with the value of the Casimir operator

$$\begin{aligned}-\frac{1}{2\langle H \rangle} &= 1 + 2\langle \vec{\mathcal{J}}_+^2 + \vec{\mathcal{J}}_-^2 \rangle = 1 + 4J(1+J), \quad J = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, \ldots\\ E_k &= -\frac{1}{2k^2}, \quad \text{multiplicity: } k^2 = (1+2J)^2 = 1, 4, 9, 16, \ldots\end{aligned}$$

The $\mathbf{SO}(4)$-representations are decomposable with respect to the position rotation $\mathbf{SO}(3)$-properties into irreducible representations of dimension $(1+2L)$ with integer $L = 0, 1, \ldots$ for $\vec{\mathcal{L}} = \vec{\mathcal{J}}_+ + \vec{\mathcal{J}}_-$

$$\begin{aligned}(2J, 2J) &\overset{\mathbf{SO}(3)}{\cong} \bigoplus_{L=0}^{2J} [2L]\\ 2J = L + N &\Rightarrow (L, N) = (2J, 0), (2J-1, 1), \ldots, (0, 2J)\end{aligned}$$

The Lenz-Runge invariance related difference $2J - L = N$ characterizing the classes $\Omega^3 \cong \mathbf{SO}(4)/\mathbf{SO}(3)$ is the radial quantum number or knot number.

1.4 Orbits of 1-Dimensional Position

In contrast to the classical framework, the time orbits in quantum theory are not valued in position space. For bound states they are valued in a Hilbert

space. In addition to the time orbits there are - in a Schrödinger picture - position orbits effected by complex valued position representations ψ - for bound states[5]

$$(t,x)\longmapsto \Psi(t,x)=e^{iEt}\psi(x)\in\mathbb{C},\quad e^{iEt}\in\mathbf{U}(1),\quad \psi\in L^2(\mathbb{R}^s)$$

In the energy eigenvalue problem of a Hamiltonian $H=\frac{p^2}{2}+V$, parametrized with 1-dimensional position translations

$$ip\cong\tfrac{d}{dx}=d_x:\quad [-\tfrac{1}{2}d_x^2+V(x)]\psi(x)=E\psi(x)$$

the real energies for the time translation eigenvalues are given in terms of real or imaginary 'momenta' for the position translation eigenvalues. The differential equation for constant potential with position translation invariance of the Hamiltonian

$$H_0=\tfrac{p^2}{2}+V_0,\quad [H_0,p]=0,\quad [d_x^2+2(E-V_0)]\psi_0(x)=0$$

is solved by two types of the representation matrix elements of the noncompact position group $\mathbb{R}$ - either with imaginary or with real eigenvalues, representing quotient groups of position $\mathbb{R}$, of the same type as for the classical time orbits in position above

$$\mathbb{R}\ni x\longmapsto\left\{\begin{array}{rl}\begin{pmatrix}\cos qx & i\sin qx\\ i\sin qx & \cos qx\end{pmatrix} & \cong\begin{pmatrix}e^{iqx} & 0\\ 0 & e^{-iqx}\end{pmatrix}\in\mathbf{SO}(2)\subset\mathbf{SU}(2)\\ E-V_0 & =\frac{q^2}{2}>0\text{ (free scattering waves)}\\ \begin{pmatrix}\cosh Qx & \sinh Qx\\ \sinh Qx & \cosh Qx\end{pmatrix} & \cong\begin{pmatrix}e^{Qx} & 0\\ 0 & e^{-Qx}\end{pmatrix}\in\mathbf{SO}_0(1,1)\subset\mathbf{SU}(1,1)\\ E-V_0 & =-\frac{Q^2}{2}<0\text{ (bound waves)}\end{array}\right.$$

Matrix elements of reducible nondecomposable representations come with strictly positive position translation powers (nildimensions) $x^N e^{\pm iqx}$ and $x^N e^{\pm Qx}$, $N=1,2,\ldots$. These representations are indefinite unitary. The order structure of the reals, i.e. the bicone property $\mathbb{R}=\mathbb{R}_+\uplus\mathbb{R}_-$, can be represented with additional factors $\vartheta(\pm x)$ and $\epsilon(x)$, e.g. in $|x|=\epsilon(x)x$.

The groups realized by time $\mathbb{R}$ and position $\mathbb{R}$ orbits are the product of $\mathbf{U}(1)$ for time with the corresponding real 1-dimensional groups for position

$$\begin{array}{rrl}H_0=\frac{p^2}{2}+V_0: & \mathbb{R}\times\mathbb{R} & \longrightarrow\mathbf{U}(1)\times\begin{pmatrix}\mathbf{SO}(2)\\ \mathbf{SO}_0(1,1)\end{pmatrix}\\ \begin{pmatrix}\text{free waves}\\ \text{bound waves}\end{pmatrix}: & (t,x) & \longmapsto e^{iEt}\begin{pmatrix}e^{\mp iqx}\\ e^{-|Qx|}\end{pmatrix},\quad E-V_0=\begin{pmatrix}\frac{q^2}{2}\\ -\frac{Q^2}{2}\end{pmatrix}\end{array}$$

The representation invariants for time (energy) and position are related to each other. The difference $E-V_0$ connects with each other the eigenvalues iE for the time translation representation with the energy E and the eigenvalues iq for the position translation representation with the momentum q. It is the nonrelativistic precursor of the relativistic energy-momentum relation $\vec{q}^{\,2}=q_0^2-m^2$ as used for quantum fields. The compact position representations in

[5] The function space $L^p_{d\mu}(S,\mathbb{C})$ will be denoted by $L^p(S)$ if the positive S-measure $d\mu$ is unique up to a factor, e.g. for a locally compact group with Haar measure.

$\mathbf{SO}(2)$ for scattering waves come for positive kinetic energies, i.e. for $E > V_0$ ($q_0^2 > m^2$) with real momentum $q^2 > 0$ ('on shell' real particles) and imaginary eigenvalue iq. The noncompact position representations in $\mathbf{SO}_0(1,1)$ for bound waves come for $E < V_0$ ($q_0^2 < m^2$) with imaginary 'momentum' $(iQ)^2 < 0$ ('off shell' virtual particles) and real eigenvalue $|Q|$.

The conjugation and, therewith, the scalar product (probability amplitude) for Hilbert spaces is determined by the positive unitary group $\mathbf{U}(1) \ni e^{iEt}$ with the time representations, not by the group with the position translation representations which for $\mathbf{U}(1)$-time representations are definite unitary $\mathbf{SU}(2)$ for free scattering waves and indefinite unitary $\mathbf{SU}(1,1)$ for bound waves.

Bound waves will be defined as square integrable functions, i.e. as elements of the position function Hilbert space. Free scattering waves are no position Hilbert space vectors, they are tempered distributions

$$\{x \longmapsto e^{-|Qx|}\} \in L^2(\mathbb{R}), \quad \{x \longmapsto e^{-iqx}\} \in \mathcal{S}'(\mathbb{R})$$

Hilbert space vectors for scattering states need square integrable momentum wave packets f in the Fourier isomorphism

$$L^2_{dp}(\mathbb{R}) \ni f \leftrightarrow \psi \in L^2_{dx}(\mathbb{R}), \quad \psi(x) = \int \frac{dp}{2\pi}\, f(p) e^{ipx}$$

Integrals without boundary go over the full integration space, here $\int dq = \int_{-\infty}^{\infty} dq$. Both the free scattering and the bound waves with compact and noncompact representation matrix elements can be Fourier expanded with the unitary position characters $e^{ipx} \in \mathbf{U}(1)$

$$\begin{aligned} e^{-|Qx|} &= \int \frac{dp}{\pi}\, \frac{|Q|}{p^2+Q^2} e^{-ipx} \\ e^{-iqx} &= \int dp\, \delta(p-q) e^{-ipx} = \oint \frac{dp}{2i\pi} \frac{1}{p-q} e^{-ipx} \end{aligned}$$

The irreducible exponentials come as residues (counterclockwise integration $\oint$) of a real momentum pole $p = q$ for the compact representations and an imaginary 'momentum' pole $p = -i|Q|$ for the noncompact ones.

1.5 Orbits of 3-Dimensional Position

The 3-dimensional position translations with rotation group $\mathbf{SO}(3)$ action are, in polar coordinates, the product $\mathbb{R}^3 \cong \Omega^2 \times \mathbb{R}_+$ of the totally ordered cone $\mathbb{R}_+$ with the radial translations and the compact 2-sphere. Both factors will be presented by corresponding orbits. The 1-dimensional case is embedded as abelian substructure

$$\mathbb{R} \subseteq \mathbf{SO}(s) \vec{\times} \mathbb{R}^s, \quad \mathbb{R}^s \cong \Omega^{s-1} \times \mathbb{R}_+, \quad s \geq 1$$

The 0-sphere consists of two points $\Omega^0 = \{\pm 1\} \cong \mathbb{I}(2)$ - forwards and backwards.

In a derivative representation, acting upon differentiable complex functions the radial and angular momentum squares look as follows

$$\begin{aligned} \vec{p}^2 &= p_r^2 + \frac{\vec{\mathcal{L}}^2}{r^2}, \quad [\vec{p}^2, \vec{\mathcal{L}}] = 0 \\ \vec{\mathcal{L}}^2 &\cong \frac{1}{\sin^2\theta}[(\sin\theta \frac{\partial}{\partial\theta})^2 + (\frac{\partial}{\partial\varphi})^2] = \frac{\partial^2}{\partial\theta^2} + \frac{1}{\tan\theta}\frac{\partial}{\partial\theta} + (\frac{1}{\sin\theta}\frac{\partial}{\partial\varphi})^2 \end{aligned}$$

The harmonic polynomials as product of spherical harmonics with the corresponding radial power are homogeneous in the coordinates $\vec{x}$. They are defined also for $\vec{x} \to 0$ in contrast to the spherical harmonics Y^L_m with the $r \to 0$ ambiguity in $\frac{\vec{x}}{r}$. The harmonic polynomials are eigenfunctions for translation invariant $\vec{p}^2$ with trivial value $\vec{q}^2 = 0$

$$(\vec{x})^L_m = r^L \mathrm{Y}^L_m(\varphi,\theta): \quad \left.\begin{array}{rl} \vec{\mathcal{L}}^2 \mathrm{Y}^L_m(\varphi,\theta) & = L(1+L)\mathrm{Y}^L_m(\varphi,\theta) \\ p_r^2 r^L & = -L(1+L)r^{L-2} \end{array}\right\} \Rightarrow \vec{\partial}^2 (\vec{x})^L_m = 0$$

For fixed L and any r they span a space acted upon with an irreducible rotation group $\mathbf{SO}(3)$-representation.

A rotation invariant Hamiltonian is decomposable into the generators H_L for each angular momentum

$$H = \bigoplus_{L=0}^{\infty} H_L \text{ with } \left\{ \begin{array}{rl} H & = \frac{\vec{p}^2}{2} + V(r), \quad [H, \vec{\mathcal{L}}] = 0 \\ H_L & = \frac{p_r^2}{2} + \frac{L(1+L)}{2r^2} + V(r) \\ & = r^n [-\frac{1}{2} d_r^2 - \frac{1+n}{r} d_r + \frac{L(1+L)-n(1+n)}{2r^2} + V(r)] \frac{1}{r^n} \\ n & = 0, \pm 1, \pm 2, \ldots \end{array}\right.$$

Attention has to be paid to the small distance $r \to 0$ behavior, prepared with the powers r^n and used below.

The radial functions $\{\psi^m_L\}$ are radial translation representation matrix elements. The noncompact representations for bound waves act on a Hilbert space, the compact ones are scattering distributions. The irreducible ones are given by

$$\mathbb{R}_+ \ni r \longmapsto \left\{ \begin{array}{lll} e^{-|Q|r} & \text{with } (d_r^2 - Q^2) e^{-|Q|r} & = 0 \\ e^{\pm i|\vec{q}|r} & \text{with } (d_r^2 + \vec{q}^2) e^{\pm i|\vec{q}|r} & = 0 \end{array}\right.$$

Their Fourier transforms involve a dipole at imaginary 'momenta' $|\vec{p}| = \pm i|Q|$ for bound states (noncompact, hyperbolic) and a pole for free scattering states (compact, spherical)

$$\begin{array}{rll} e^{-|Q|r} & = \int \frac{d^3p}{\pi^2} \frac{|Q|}{(\vec{p}^2+Q^2)^2} e^{-i\vec{p}\vec{x}} & \text{from } L^2(\mathbb{R}^3) \\ \frac{\sin|\vec{q}|r}{|\vec{q}|r} = \int \frac{d^3p}{2\pi|\vec{q}|} \delta(\vec{p}^2 - \vec{q}^2) e^{-i\vec{p}\vec{x}} & = \oint \frac{d^3p}{4i\pi^2|\vec{q}|} \frac{1}{\vec{p}^2-\vec{q}^2} e^{-i\vec{p}\vec{x}} & \text{from } \mathcal{S}'(\mathbb{R}^3) \end{array}$$

With the integral around all poles

$$a \in \mathbb{R}: \quad \delta(a) \to \oint \frac{1}{2i\pi} \frac{1}{a}$$

the representations can be written as residues at the negative and positive invariants.

1.5.1 Scattering Distributions

Compact radial representations are spread to the 2-sphere with a factor $\frac{1}{r}$ and the separation of the spherical harmonics

$$\psi(\vec{x}) = \sum_{L=0}^{\infty} \sum_{m=-L}^{L} (\tfrac{\vec{x}}{r})^L_m \frac{D_L(r)}{r} \quad \Rightarrow [d_r^2 - \frac{L(1+L)}{r^2} + 2(E - V(r))] D_L(r) = 0$$
$$D_L(r) = r\psi_L(r)$$

An irreducible compact radial position representation requires a constant potential

$$D_0(r) = e^{\pm i|\vec{q}|r} \Rightarrow H = \tfrac{\vec{p}^2}{2} + V_0, \quad E - V_0 = \tfrac{\vec{q}^2}{2}$$

The solutions for general L use the spherical Bessel functions

$$[d_r^2 - \tfrac{L(1+L)}{r^2} + \vec{q}^2]D_L(r) = 0 \Rightarrow D_L(r) = |\vec{q}|r j_L(|\vec{q}|r)$$

which have to be considered as tempered distributions, e.g.

$$\begin{pmatrix} j_0(r) \\ \frac{\vec{x}}{r} j_1(r) \end{pmatrix} = \begin{pmatrix} \frac{\sin r}{r} \\ \frac{\vec{x}}{r} \frac{\sin r - r\cos r}{r^2} \end{pmatrix} = \int \tfrac{d^3 p}{2\pi} \begin{pmatrix} 1 \\ i\vec{p} \end{pmatrix} \delta(\vec{p}^2 - 1) e^{-i\vec{p}\vec{x}}$$

They act via Fourier-Bessel transforms $\int_0^\infty dq f_L(q) q j_L(qr)$ upon momentum wave packets $f_L \in L^2(\mathbb{R}_+)$.

1.5.2 Multipole Energy Measures for the Kepler Potential

Noncompact radial representations come with the separation of the harmonic polynomials for the irreducible $\mathbf{SO}(3)$-representations. This leads to the Schrödinger equations for the position representation matrix elements

$$\psi(\vec{x}) = \sum_{L=0}^{\infty} \sum_{m=-L}^{L} (\vec{x})^L_m d_L(r) \quad \Rightarrow [d_r^2 + \tfrac{2(1+L)}{r} d_r + 2(E - V(r))] d_L(r) = 0$$
$$r = |\vec{x}|$$

For an irreducible noncompact radial position representation as bound solution an attractive Kepler potential is necessary with an associate angular momentum dependent momentum unit - a 'ground state' for each angular momentum

$$d_L(r) = e^{-|Q|r} \Rightarrow V(r) = -\tfrac{1}{r} \Rightarrow (1+L)Q_L = (1+L)\sqrt{-2E_{L0}} = 1$$

There arises the quantum analogue to the classical parameter $L\sqrt{-2E}$. For the attractive Kepler potential and negative energy the imaginary radial 'momentum' $\sqrt{2E}$ is 'quantized' (integer wave numbers k) in the radial bound waves

$$\begin{array}{r} [d_r^2 + \frac{2(1+L)}{r} d_r + 2(E + \frac{1}{r})] d_L(r) = 0 \\ \frac{\rho(r)}{2} = |Q|r \\ E = -\frac{Q^2}{2} < 0 \end{array} \Rightarrow \left\{ \begin{array}{rl} d_L(r) & = \mathrm{L}^N_{1+2L}(\rho) e^{-\frac{\rho}{2}} \\ \frac{1}{Q_k} = k & = 1 + 2J = 1 + L + N \\ & \quad L, N = 0, 1, \ldots \\ \psi^{2J}_{Lm}(\vec{x}) & \sim (\frac{\vec{x}}{k})^L_m \, \mathrm{L}^N_{1+2L}(\frac{2r}{k}) \, e^{-\frac{r}{k}} \end{array} \right.$$

The $\mathbf{SO}(4)$ multiplets comprise all wave functions ψ^{2J} with equal sum $L + N = 2J$ for the principal quantum number $k = 1 + 2J$ with angular momentum $(1 + 2L)$-multiplets for $\mathbf{SO}(3)$ and radial quantum numbers N for Lenz-Runge classes $\Omega^3 \cong \mathbf{SO}(4)/\mathbf{SO}(3)$. The products of harmonic and Laguerre polynomials are related to the Ω^3-analogue of the spherical harmonics

on Ω^2. The Fourier transformations of the bound state wave functions involve the $\mathbf{SO}(3)$-invariant momentum measure $\frac{d^3p}{4\pi^2}(\frac{2}{1+\vec p^2})^2$ with dipoles at the negative invariants (imaginary 'momentum' eigenvalues)

$$Q=1: \quad \psi_0^0(\vec x)\sim e^{-Qr}=\int \frac{d^3p}{Q^3 4\pi^2}(\frac{2Q^2}{Q^2+\vec p^2})^2 e^{-i\vec p\vec x}=\int \frac{d^3p}{4\pi^2}(\frac{2}{1+\vec p^2})^2 e^{-i\vec pQ\vec x}$$

The $k=2$ bound states quartet have tripole vector measure

$$\begin{aligned} Q=\tfrac12: \quad \begin{pmatrix}\psi_0^1\\ \psi_1^1\end{pmatrix}(\vec x)\sim \begin{pmatrix}\frac14 \mathrm{L}_1^1(2Qr)\\ \frac{Q\vec x}{2}\,\mathrm{L}_2^0(2Qr)\end{pmatrix} &= \begin{pmatrix}\frac{1-Qr}{2}\\ \frac{Q\vec x}{2}\end{pmatrix} e^{-Qr}\\ &=\int \frac{d^3p}{4\pi^2}(\frac{2}{1+\vec p^2})^3 e^{-i\vec pQ\vec x}\begin{pmatrix}\frac{\vec p^2-1}{2}\\ i\vec p\end{pmatrix}\end{aligned}$$

The normalized 4-vector on the 3-sphere under the integral

$$\frac{2}{1+\vec p^2}\begin{pmatrix}\frac{\vec p^2-1}{2}\\ i\vec p\end{pmatrix}=\begin{pmatrix}q_0\\ i\vec q\end{pmatrix}=\begin{pmatrix}\cos\chi\\ i\sin\chi\frac{\vec p}{|\vec p|}\end{pmatrix}\in\Omega^3 \text{ with } q^2=q_0^2+\vec q^2=1$$

is the analogue $\mathrm{Y}^{(1,1)}(q)\sim\begin{pmatrix}q_0\\ i\vec q\end{pmatrix}\in\Omega^3$ to the normalized 3-vector $\mathrm{Y}^1(\frac{\vec p}{|\vec p|})\sim\frac{\vec p}{|\vec p|}\in\Omega^2$ which builds the 2-sphere harmonics $\mathrm{Y}^L(\frac{\vec p}{|\vec p|})\sim(\frac{\vec p}{|\vec p|})^L$. It depends on three angles for Ω^3 which can be parametrized with a 3-vector $\vec p\in\mathbb{R}^3$. The higher order Ω^3-harmonics arise from the totally symmetric traceless products $\mathrm{Y}^{(2J,2J)}(q)\sim(q)^{2J}$, e.g. the nine independent components in the (4×4)-matrix

$$\begin{aligned}\mathrm{Y}^{(2,2)}(q) &\sim (q)^2_{jk}=q_jq_k-\frac{\delta_{jk}}{4}\cong\left(\begin{array}{c|c}\frac{3q_0^2-\vec q^2}{4} & iq_0q_a\\ \hline iq_0q_b & q_aq_b-\frac{\delta_{ab}}{4}\end{array}\right)\\ q_aq_b-\frac{\delta_{ab}}{4} &= q_aq_b-\frac{\delta_{ab}}{3}\vec q^2-\frac{\delta_{ab}}{3}\frac{3q_0^2-\vec q^2}{4} \text{ with } q^2=1\end{aligned}$$

which are used for the $k=3$ bound states nonet with quadrupole tensor measure

$$\begin{aligned} Q=\tfrac13: \quad \begin{pmatrix}\psi_0^2\\ \psi_1^2\\ \psi_2^2\end{pmatrix}(\vec x) &\sim \begin{pmatrix}\frac13\mathrm{L}_1^2(2Qr)\\ \frac{Q\vec x}{2}\,\frac16\mathrm{L}_3^1(2Qr)\\ \frac{Q^2}{2}(\mathbf{1}_3\frac{r^2}{3}-\vec x\otimes\vec x)\mathrm{L}_5^0(2Qr)\end{pmatrix}=\begin{pmatrix}1-2Qr+\frac{2Q^2r^2}{3}\\ \frac{2-Qr}{3}\frac{Q\vec x}{2}\\ \frac{Q^2}{2}(\mathbf{1}_3\frac{r^2}{3}-\vec x\otimes\vec x)\end{pmatrix}e^{-Qr}\\ &=\int\frac{d^3p}{4\pi^2}(\frac{2}{1+\vec p^2})^4 e^{-i\vec pQ\vec x}\begin{pmatrix}3(\frac{\vec p^2-1}{2})^2-\vec p^2\\ i\vec p\,\frac{\vec p^2-1}{2}\\ 3\vec p\otimes\vec p-\mathbf{1}_3\vec p^2\end{pmatrix}\\ &=\int\frac{d^3p}{4\pi^2}(\frac{2}{1+\vec p^2})^2 e^{-i\vec pQ\vec x}\begin{pmatrix}3q_0^2-\vec q^2\\ iq_0\vec q\\ 3\vec q\otimes\vec q-\mathbf{1}_3\vec q^2\end{pmatrix}\end{aligned}$$

and in general with $2J$-dependent multipole measure of the momenta

$$Q=\frac{1}{1+2J},\ J=L+N: \quad \psi_L^{2J}(\vec x)\sim\int\frac{d^3p}{4\pi^2}(\frac{2}{1+\vec p^2})^2 e^{-i\vec pQ\vec x}(q)^{2J},\quad q=\frac{2}{1+\vec p^2}\begin{pmatrix}\frac{\vec p^2-1}{2}\\ i\vec p\end{pmatrix}$$

1.5.3 Nonrelativistic Color Symmetry

Separating the harmonic polynomials with a squared radial dependence, there arise the radial equations

$$\begin{aligned}\psi(\vec x)=\sum_{L=0}^{\infty}\sum_{m=-L}^{L}(\vec x)^L_m\Delta_L(\rho) \quad &\Rightarrow [\rho d_\rho^2+(\tfrac32+L-\rho)d_\rho+\tfrac{E-V(r)}{2}]\Delta_L(\rho)=0\\ \rho=r^2=\vec x^2 &\end{aligned}$$

An irreducible exponential with squared radial dependence as solution determines the harmonic oscillator potential, normalized[6] with a momentum unit $|Q|$

$$\Delta_L(r^2) = e^{-\frac{Q^2 r^2}{2}} \Rightarrow V(r) = \frac{(Q_L r)^2}{2} Q_L^2, \quad E = (\tfrac{3}{2} + L) Q_L^2$$

The momentum unit can be chosen L-independent, e.g. $Q_L = 1$. The general solutions come with Laguerre polynomials of degree N and - in contrast to the Kepler bound state solutions with principal quantum number dependent exponentials $e^{-\frac{r}{k}}$ - with one exponential only

$$[d_r^2 + \tfrac{2(1+L)}{r} d_r + 2(E - \tfrac{r^2}{2})]\Delta_L(r^2) = 0 \Rightarrow \left\{ \begin{array}{rl} \Delta_L(r^2) & = L^N_{\frac{1+2L}{2}}(r^2) e^{-\frac{r^2}{2}} \\ E_k & = \frac{3}{2} + k = \frac{3}{2} + L + 2N \\ & \qquad L, N = 0, 1, \ldots \\ \psi^k_{Lm}(\vec{x}) & \sim (\vec{x})^L_m \; \mathrm{L}^N_{\frac{1+2L}{2}}(r^2) \; e^{-\frac{r^2}{2}} \end{array} \right.$$

The harmonic oscillator Hamiltonian

$$H = \frac{\vec{p}^2 + \vec{x}^2}{2} = \frac{\{\mathrm{u}^a, \mathrm{u}^\star_a\}}{2} \text{ with creation operators } \mathrm{u}^a = \frac{x^a - ip^a}{\sqrt{2}}, \quad [\mathrm{u}^\star_b, \mathrm{u}^a] = \delta^b_a$$

generates time orbits of the Hilbert vectors with k quanta. They have a Schrödinger representation as position orbits

$$\begin{array}{rl} |a_1, \ldots a_k\rangle & = \frac{\mathrm{u}^{a_1} \cdots \mathrm{u}^{a_k}}{\sqrt{k!}} |0\rangle \cong \{\vec{x} \longmapsto x^{a_1} \cdots x^{a_k} e^{-\frac{r^2}{2}}\} \in L^2(\mathbb{R}^3) \\ \mathrm{u}(t)^a & = e^{it} \mathrm{u}^a, \quad |a_1, \ldots a_k\rangle(t) = e^{ikt} |a_1, \ldots a_k\rangle \end{array}$$

The complex representation of the three position translations $\mathbb{R}^3 \hookrightarrow \mathbb{C}^3$ leads to a color $\mathbf{SU}(3)$-invariance - with Gell-Mann matrices

$$\chi_A \lambda^A = \begin{pmatrix} \chi_3 + \frac{\chi_8}{\sqrt{3}} & \chi_1 - i\chi_2 & \chi_4 - i\chi_5 \\ \chi_1 + i\chi_2 & -\chi_3 + \frac{\chi_8}{\sqrt{3}} & \chi_6 - i\chi_7 \\ \chi_4 + i\chi_5 & \chi_6 + i\chi_7 & -2\frac{\chi_8}{\sqrt{3}} \end{pmatrix}, \quad \mathcal{C} = \tfrac{i}{2} \mathrm{u}^a \lambda^b_a \mathrm{u}^\star_b, \quad [\mathcal{C}, H] = 0$$

The $\mathbf{SU}(3)$-representations $[2C_1, 2C_2]$ are characterized by two integers, they have the dimension

$$\dim_{\mathbb{C}}[2C_1, 2C_2] = (1 + 2C_1)(1 + 2C_2)(1 + C_1 + C_2)$$

The harmonic oscillator representations $[k, 0]$ (singlet, triplet, sextet, etc.) are the totally symmetric products of $\mathbf{SU}(3)$-triplets $[1, 0]$

$$\bigvee^k \mathbb{C}^3 \cong \mathbb{C}^{\binom{2+k}{2}}, \quad \dim_{\mathbb{C}}[k, 0] = \binom{2+k}{k} = 1, 3, 6, \ldots, \quad k = 0, 1, 2, \ldots$$

The rotation group, generated by the transposition antisymmetric Lie subalgebra

$$\mathbf{SO}(3) \hookrightarrow \mathbf{SU}(3) \text{ with } \left\{ \begin{array}{l} \chi_A i \frac{\lambda^A - (\lambda^A)^T}{2} = \begin{pmatrix} 0 & \chi_2 & \chi_5 \\ -\chi_2 & 0 & \chi_7 \\ -\chi_5 & -\chi_7 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \varphi_3 & -\varphi_2 \\ -\varphi_3 & 0 & \varphi_1 \\ \varphi_2 & -\varphi_1 & 0 \end{pmatrix} \\ \mathcal{L}^a = \epsilon^{abc} \mathrm{u}^b \mathrm{u}^\star_c, \quad (\mathcal{L}^1, \mathcal{L}^2, \mathcal{L}^3) = (\mathcal{C}^7, -\mathcal{C}^5, \mathcal{C}^2) \end{array} \right.$$

[6] Usually, the additive term $V_0 = \frac{3}{2} Q^2$ is introduced as ground state energy.

comes with the real 5-dimensional orientation manifold, i.e. the rotation group orbits $\mathbf{SU}(3)/\mathbf{SO}(3)$ in the color group. which describes the $\binom{4}{2}-1$ relative phases of the three position directions in complex quantum structures. The principal quantum number $k = L + 2N$ is the sum of the angular momentum quantum number L for $\mathbf{SO}(3)$ and the radial quantum number (knot number) N for the rotation group classes in $\mathbf{SU}(3) \cong \mathbf{SO}(3) \times \mathbf{SU}(3)/\mathbf{SO}(3)$. One has with the angular momentum degeneracy $1 + 2L$ the energy degeneracy given by the dimensions of the $\mathbf{SU}(3)$-representations

$$\text{multiplicity } \binom{k+2}{2}, \quad [k,0] \overset{\mathbf{SO}(3)}{\cong} \begin{cases} \bigoplus_{L=0}^{k}[2L], & k = 0, 2, \dots \text{ (even)} \\ \bigoplus_{L=1}^{k}[2L], & k = 1, 3, \dots \text{ (odd)} \end{cases}$$

$$E_{LN} - \tfrac{3}{2} = L + 2N = k \Rightarrow (L, N) = \begin{cases} (k,0), (k-2,1), \dots, (0, \frac{k}{2}) \\ (k,0), (k-2,1), \dots, (1, \frac{k-1}{2}) \end{cases}$$

Chapter 2

Free Particles as Translation Orbits

Free particle properties are determined by the Poincaré group and its irreducible Hilbert representations[28]. Compact time representation matrix elements $t \longmapsto e^{iEt}$, multiplied either with compact position representation matrix elements, e.g. $(t,\vec{x}) \longmapsto e^{iEt}\ e^{-i\vec{q}\vec{x}}$ and $(t,\vec{x}) \longmapsto e^{iEt}\ \frac{\sin|\vec{q}|r}{r}$, or with noncompact ones, e.g. $(t,\vec{x}) \longmapsto e^{iEt}\ e^{-|Q|r}$, are Lorentz compatibly connected in Feynman propagators of particle fields for position dimension $s = 1, 2, 3, \ldots$

$$\frac{i}{\pi}\int \frac{d^{1+s}q}{(2\pi)^s}\frac{1}{q^2+io-m^2}e^{iqx} \text{ with } \frac{i}{\pi}\frac{1}{q^2+io-m^2} = \delta(q^2-m^2) + \frac{i}{\pi}\frac{1}{q_{\rm P}^2-m^2}$$

They involve as energy-momentum measures the Dirac distribution and the principal value pole distribution with $q_{\rm P}^2$.

2.1 Representations of Cartan Spacetime

2-dimensional spacetime translations without rotation degrees of freedom are acted upon with the orthochronous abelian Lorentz group - they constitute Cartan spacetime

$$\mathbf{SO}_0(1,1)\ \vec{\times}\ \mathbb{R}^2$$

The nonrelativistic free scattering waves are embedded with compact translation representations $x \longmapsto e^{iqx}$, integrated with $\mathbf{SO}_0(1,1)$-invariant measures of the momenta or of the kinetic energies over the threshold

$$\int d^2q\ \delta(q_0^2 - q_3^2 - m^2) \Rightarrow \left\{ \begin{array}{ll} \frac{dq_3}{q_0} & \text{with } q_0 = \sqrt{q_3^2+m^2} \\ \frac{dq_0\vartheta(q_0^2-m^2)}{q_3} & \text{with } q_3 = \sqrt{q_0^2-m^2} \end{array} \right.$$

This leads to the on-shell propagator contribution with compact representation matrix elements of the translation group $\mathbb{R}^2$

$$\mathbb{R}^2 \ni x = (t,z) \longmapsto \int d^2q\ \delta(q^2-m^2)e^{iqx} = \left\{ \begin{array}{l} \int \frac{dq_3}{q_0}\ e^{-iq_3z}\ \cos q_0 t \\ \int \frac{dq_0\vartheta(q_0^2-m^2)}{|q_3|}\ e^{iq_0t}\ \cos q_3 z \end{array} \right.$$

The indefinite unitary (noncompact) representations ('off shell') of the translations embed the bound waves. They come in Fourier transformed principal value distributions of energy-momenta and are integrated with the invariant measure for binding energies below the threshold

$$\frac{dq_0\vartheta(m^2-q_0^2)}{|Q|} \text{ with } |Q| = \sqrt{m^2-q_0^2}$$

There arise both contributions from compact and noncompact position representations ('on shell' and 'off shell' with real and imaginary 'momenta' resp.)

$$\begin{aligned} \mathbb{R}^2 \ni x \quad &\longmapsto \int d^2q \, \frac{1}{q_{\mathrm{P}}^2-m^2}e^{iqx} \\ &= \begin{cases} -\pi \int \frac{dq_3}{q_0} \; e^{-iq_3z} \sin q_0|t| \\ \pi \int dq_0 e^{iq_0t}[\frac{\vartheta(q_0^2-m^2)}{|q_3|} \; \sin|q_3z| - \frac{\vartheta(m^2-q_0^2)}{|Q|} \; e^{-|Qz|}] \end{cases} \end{aligned}$$

2.2 Representations of Minkowski Spacetime

4-dimensional spacetime translations[36] with nontrivial rotation degrees of freedom come with the action of the orthochronous Lorentz group

$$\mathbf{SO}_0(1,3) \; \vec{\times} \, \mathbb{R}^4$$

The Lorentz invariant measures for momenta and energies (kinetic and binding) involve the 2-sphere measure $\int d\Omega^2 = \int_0^{2\pi} d\varphi \int_{-1}^1 d\cos\theta$

$$\begin{aligned} \frac{d^3q}{q_0} = \frac{q^2\vartheta(q)dq}{q_0} \; d\Omega^2 &\quad \text{with } q_0 = \sqrt{\vec{q}^2+m^2} \\ dq_0\vartheta(q_0^2-m^2)|\vec{q}| \; d\Omega^2 &\quad \text{with } |\vec{q}| = \sqrt{q_0^2-m^2} \\ dq_0\vartheta(m^2-q_0^2)|Q| \; d\Omega^2 &\quad \text{with } |Q| = \sqrt{m^2-q_0^2} \end{aligned}$$

The $(r=0)$ regular spherical Bessel function $j_0(r) = \frac{\sin r}{r}$ is a compact position representation matrix element spread over the 2-sphere with the Kepler factor $\frac{1}{r}$

$$\mathbb{R}^4 \ni x = (t,\vec{x}) \longmapsto \int d^4q \; \delta(q^2-m^2)e^{iqx} = \begin{cases} \int \frac{d^3q}{q_0} \; e^{-i\vec{q}\vec{x}} \; \cos q_0t \\ 2\pi \int dq_0\vartheta(q_0^2-m^2) \; e^{iq_0t} \; \frac{\sin|\vec{q}|r}{r} \end{cases}$$

For the indefinite unitary (noncompact) position representation (imaginary 'momenta') the additional rotation degrees of freedom for the nonabelian group $\mathbf{SO}(3)$ change the situation in comparison to Cartan spacetime: The simple principal value pole does not yield a position representation matrix element, as seen, e.g., in the $(r=0)$-singular Yukawa potential $\frac{e^{-|Q|r}}{r}$

$$\mathbb{R}^4 \ni x \longmapsto -\int d^4q \; \frac{1}{q_{\mathrm{P}}^2-m^2}e^{iqx} = \begin{cases} \pi \int \frac{d^3q}{q_0} \; e^{-i\vec{q}\vec{x}} \sin q_0|t| \\ 2\pi^2 \int dq_0 e^{iq_0t} \frac{\vartheta(q_0^2-m^2)\cos|\vec{q}|r+\vartheta(m^2-q_0^2)e^{-|Q|r}}{r} \end{cases}$$

Position representation matrix elements have to be regular at $r = 0$. They start with a dipole

$$\mathbb{R}^4 \ni x \longmapsto \int d^4q \, \frac{1}{(q_{\rm P}^2-m^2)^2} e^{iqx} = \left\{ \begin{array}{l} -\pi \int \frac{d^3q}{q_0} \, e^{-i\vec q\vec x} \, \frac{\sin q_0|t| - q_0|t| \cos q_0 t}{2q_0^2} \\ \pi^2 \int dq_0 e^{iq_0 t} [\frac{\vartheta(q_0^2-m^2)}{|\vec q|} \sin |\vec q| r + \frac{\vartheta(m^2-q_0^2)}{|Q|} e^{-|Q| r}] \end{array} \right.$$

Dipoles do not occur in the Feynman propagators for particles. The connection between spacetime dimensionality and higher order poles will be discussed in the overnext section.

2.3 Free Particles as Orbits of Flat Spacetime

Free particles, as defined by Wigner[28], are acted upon with Hilbert space representation of the semidirect (covering) Poincaré group

$$\text{symmetry for free particle fields: } \mathbf{U}(1) \times \mathbf{SL}(\mathbb{C}^2) \ \vec{\times}\, \mathbb{R}^4$$

with the homogeneous internal phase group $\mathbf{U}(1)$, e.g. the electromagnetic group, the Lorentz (cover) group $\mathbf{SL}(\mathbb{C}^2)$ and the inhomogeneous spacetime translations. They have a translation invariant m^2 (mass), a rotation invariant J ($\mathbf{SU}(2)$-spin for $m^2 > 0$ or $\mathbf{SO}(2)$-polarization for $m^2 = 0$) and an internal phase number, e.g. an electromagnetic charge number. These properties determine the Poincaré group representations

$$\int d^4q \ (q)^{2J} \delta(q^2 - m^2) e^{iqx} \text{ with } J \in \{0, \tfrac{1}{2}, 1, \dots\}$$

For nontrivial angular momentum, mass and momentum have to be embedded in Lorentz vectors, starting with the two Weyl representations, left and right handed

$$(m, \pm\vec q) \hookrightarrow (q, \hat q) = q_0 \pm \vec q = \begin{pmatrix} q_0 \pm q_3 & \pm(q_1 - iq_2) \\ \pm(q_1 + iq_2) & q_0 \mp q_3 \end{pmatrix} \text{ with } q^2 = m^2$$

The representation matrix elements of the Poincaré group are Fock state expectation values, e.g. of a free scalar Bose field $\mathbf{\Phi}$ (like a neutral stable pion) or of a free massive spin $\frac{1}{2}$ Fermi field $\mathbf{\Psi}$ (like an electron-positron)

$$\begin{array}{ll} \langle\{\mathbf{\Phi}(z), \mathbf{\Phi}(y)\}\rangle & = \int \frac{d^4q}{(2\pi)^3} \qquad m \, \delta(q^2 - m^2) e^{iqx} \quad \text{with } x = y - z \\ \langle[\mathbf{\Psi}(z), \overline{\mathbf{\Psi}}(y)]\rangle & = \int \frac{d^4q}{(2\pi)^3} (\gamma q + m) \delta(q^2 - m^2) e^{iqx} \end{array}$$

The irreducible Hilbert space representations of the Poincaré group are induced[28] (more below for the general inducing procedure) by representations of the direct product subgroups with the compact fixgroups ('little groups') for the energy-momenta

$$\text{symmetry of free particles: } \left\{ \begin{array}{ll} \mathbf{U}(1) \times \mathbf{SU}(2) \times \mathbb{R}^4, & m^2 > 0 \\ \mathbf{U}(1) \times \mathbf{SO}(2) \times \mathbb{R}^4, & m^2 = 0 \end{array} \right.$$

The groups act upon the creation and annihilation operators for particles $(\mathrm{u}(\vec q), \mathrm{u}^\star(\vec q))$, e.g. for the neutral pion field as the direct integral with Lorentz invariant measure of the boosts $\mathbf{SL}(\mathbb{C}^2)/\mathbf{SU}(2) \cong \mathcal{Y}^3 \cong \mathbb{R}^3$

$$\mathbf{\Phi} = \sqrt{2m} \;{}^{\oplus}\!\!\int_{\mathbf{R}^3} \frac{d^3q}{2q_0(2\pi)^3} \frac{\mathrm{u}(\vec q)+\mathrm{u}^\star(\vec q)}{\sqrt 2} \text{ with } q_0 = \sqrt{m^2+\vec q^2}$$

For nontrivial rotation properties the creation and annihilation operators with rotation group action are embedded into the particle fields with Lorentz group action by representations of the symmetric spaces involved, i.e. the boosts $\mathbf{SL}(\mathbb{C}^2)/\mathbf{SU}(2) \cong \mathcal{Y}^3$ for massive and $\mathbf{SL}(\mathbb{C}^2)/\mathbf{SO}(2) \cong \mathcal{Y}^3 \times \Omega^2$ for massless particles. E.g. the electron with antiparticles $(\mathrm{a}(\vec q), \mathrm{a}^\star(\vec q))$ (positron) is acted upon with the left and right-handed Weyl representations $(s, \hat s)$ of the boosts

$$\begin{aligned}\mathbf{\Psi} &= \begin{pmatrix}\mathbf{l}^{\dot A}\\ \mathbf{r}^{A}\end{pmatrix} = \sqrt{2m} \;{}^{\oplus}\!\!\int_{\mathbf{R}^3} \frac{d^3q}{2q_0(2\pi)^3} \begin{pmatrix} s(\frac{q}{m})^{\dot A}_a \frac{\mathrm{u}(\vec q)^a+\mathrm{a}^\star(\vec q)^a}{\sqrt 2} \\ \hat s(\frac{q}{m})^{A}_a \frac{\mathrm{u}(\vec q)^a-\mathrm{a}^\star(\vec q)^a}{\sqrt 2}\end{pmatrix}\\
\text{with} &\left\{\begin{array}{rl} (s,\hat s)(\frac{q}{m}) = & \sqrt{\frac{m+q_0}{2m}}(\mathbf{1}_2 \pm \frac{\vec q}{m+q_0})\\ & = e^{\pm\frac{\vec\psi}{2}},\ \vec\psi = \frac{\vec q}{|\vec q|}\mathrm{artanh}\frac{|\vec q|}{q_0}\\ \text{spin } \mathbf{SU}(2): & a = 1,2\\ \text{Lorentz } \mathbf{SL}(\mathbb{C}^2): & A, \dot A = 1,2\end{array}\right.\end{aligned}$$

The translation orbits give the free particle fields on spacetime, e.g.

$$\mathbb{R}^4 \ni x \longmapsto \mathbf{\Phi}(x) = \sqrt{2m} \;{}^{\oplus}\!\!\int_{\mathbf{R}^3} \frac{d^3q}{2q_0(2\pi)^3} \frac{e^{ipx}\mathrm{u}(\vec q)+e^{-ipx}\mathrm{u}^\star(\vec q)}{\sqrt 2}$$

With the exception of an abelian phase group $\mathbf{U}(1)$, free particles have only spacetime related symmetries. Strictly speaking, free particles have no color symmetry, no isopin symmetry[42]. This is seen, e.g. for isospin, in the different masses of proton and neutron (always as free particles) which is implemented in the standard model of electroweak and strong interactions[27] by the ground state degeneracy: Hyperisospin, i.e. hypercharge-isospin, $\mathbf{U}(2)$-symmetry for the interactions is 'bleached' to the electromagnetic $\mathbf{U}(1)$-symmetry for free particles; isospin disappears as symmetry, the multiplicities remain. Color $\mathbf{SU}(3)$-symmetry for the strong interactions is postulated to be even confined for free particles, there remain only color trivial singlets.

2.4 Multipole Energy-Momentum Measures

In Feynman propagators there arise measures for spheres $\Omega^s \cong \mathbf{SO}(1+s)/\mathbf{SO}(s)$, $s = 1, 2\ldots$ and hyperboloids $\mathcal{Y}^s \cong \mathbf{SO}_0(1,s)/\mathbf{SO}(s)$. They can be constructed with the defining representations of $\mathbf{SO}(1+s)$ and $\mathbf{SO}_0(1,s)$ acting on energy-momenta (imaginary 'momenta' for spheres)

$$\begin{array}{rl}\log \mathbf{SO}(1+s) \;\vec\oplus\; \mathbb{R}^{1+s}: & \left(\begin{array}{c|c} 0 & i\vec\chi\\ \hline i\vec\chi & \log\mathbf{SO}(s)\end{array}\right)\begin{pmatrix} q_0\\ \hline i\vec q\end{pmatrix}\\ & q^2 = q_0^2 - (i\vec q)^2 = q_0^2 + \vec q^2\\ \log \mathbf{SO}(1,s) \;\vec\oplus\; \mathbb{R}^{1+s}: & \left(\begin{array}{c|c} 0 & \vec\psi\\ \hline \vec\psi & \log\mathbf{SO}(s)\end{array}\right)\begin{pmatrix} q_0\\ \hline \vec q\end{pmatrix}\\ & q^2 = q_0^2 - \vec q^2\end{array}$$

Positive vectors $q^2 > 0$ have $\mathbf{SO}(s)$ as fixgroup. The representations of the fixgroup classes are parametrizable by unit vectors

$$q^2 = 1 : \quad \begin{cases} \begin{pmatrix} q_0 \\ \hline i\vec q \end{pmatrix} &= \begin{pmatrix} \cos\chi \\ \frac{\vec q}{|\vec q|} i \sin\chi \end{pmatrix}, \quad \left(\begin{array}{c|c} q_0 & iq_a \\ \hline iq_b & \delta_{ab} - \frac{q_a q_b}{1+q_0} \end{array}\right) \quad \in \mathbf{SO}(1+s)/\mathbf{SO}(s) \\ \begin{pmatrix} q_0 \\ \hline \vec q \end{pmatrix} &= \begin{pmatrix} \cosh\psi \\ \frac{\vec q}{|\vec q|} \sinh\psi \end{pmatrix}, \quad \left(\begin{array}{c|c} q_0 & q_a \\ \hline q_b & \delta_{ab} + \frac{q_a q_b}{1+q_0} \end{array}\right) \quad \in \mathbf{SO}_0(1,s)/\mathbf{SO}(s) \end{cases}$$

Unit vectors $q^2 = 1$ can be used to parametrize the positive $\mathbf{SO}(1+s)$ and $\mathbf{SO}_0(1,s)$-invariant measures, unique up to a factor. In addition, there are other parametrizations, both with a finite and infinite range - with an trigonometric angle χ or hyperbolic 'angle' ψ, with imaginary 'momenta' ip and real momenta p and with imaginary Riemann parameters iv for spheres and real Poincaré parameters v for hyperboloids.

The parametrizations for 1-sphere and 1-hyperboloid are

$$\begin{array}{llll} \Omega^1 \ni \begin{pmatrix} \cos\chi \\ i\sin\chi \end{pmatrix} & = \begin{pmatrix} q_0 \\ iq \end{pmatrix} & = \frac{1}{\sqrt{1+p^2}} \begin{pmatrix} 1 \\ ip \end{pmatrix} & = \frac{2}{1+v^2} \begin{pmatrix} \frac{1-v^2}{2} \\ iv \end{pmatrix} \\ \mathcal{Y}^1 \ni \begin{pmatrix} \cosh\psi \\ i\sinh\psi \end{pmatrix} & = \begin{pmatrix} |q_0| \\ q \end{pmatrix} & = \frac{1}{\sqrt{1-p^2}} \begin{pmatrix} 1 \\ p \end{pmatrix} & = \frac{2}{1-v^2} \begin{pmatrix} \frac{1+v^2}{2} \\ v \end{pmatrix} \end{array}$$

Sphere parametrizations with a square root give the half sphere only. From the 1-forms

$$\begin{array}{llll} \begin{pmatrix} -\sin\chi d\chi \\ i\cos\chi d\chi \end{pmatrix} & = \begin{pmatrix} dq_0 \\ idq \end{pmatrix} & = \frac{1}{\sqrt{1+p^2}^3} \begin{pmatrix} -pdp \\ idp \end{pmatrix} & = \frac{1}{(1+v^2)^2} \begin{pmatrix} -4vdv \\ i2(1-v^2)dv \end{pmatrix} \\ \begin{pmatrix} \sinh\psi d\psi \\ \cos\psi d\psi \end{pmatrix} & = \begin{pmatrix} d|q_0| \\ dq \end{pmatrix} & = \frac{1}{\sqrt{1-p^2}^3} \begin{pmatrix} pdp \\ dp \end{pmatrix} & = \frac{1}{(1-v^2)^2} \begin{pmatrix} 4vdv \\ 2(1+v^2)dv \end{pmatrix} \end{array}$$

one obtains the measures

$$\begin{array}{llll} \int d\Omega^1 & = \int_{-\pi}^{\pi} d\chi & = \int_{-1}^{1} \frac{dq_0}{\sqrt{1-q_0^2}} & = \int_{-1}^{1} \frac{dq}{\sqrt{1-q^2}} \\ & & = \int_{-\infty}^{\infty} \frac{2dp}{1+p^2} & = \int_{-\infty}^{\infty} \frac{2dv}{1+v^2} = \oint_{p=i} \frac{2dp}{1+p^2} = 2\pi \\ \int d\mathcal{Y}^1 & = \int_{-\infty}^{\infty} d\psi & = \int_{1}^{\infty} \frac{dq_0}{\sqrt{q_0^2-1}} & = \int_{-\infty}^{\infty} \frac{dq}{2\sqrt{1+q^2}} \\ & & = \int_{-1}^{1} \frac{dp}{1-p^2} & = \int_{-1}^{1} \frac{dv}{1-v^2} \end{array}$$

The v-parametrization coincides with the momentum p-parametrization only for $s = 1$.

The abelian case $s = 1$ is embedded in the general case with the $(s-1)$-sphere and factors q^{s-1} with $|\vec q| = q$

$$\begin{aligned} \int \begin{pmatrix} d\Omega^s \\ d\mathcal{Y}^s \end{pmatrix} &= \int d^{1+s}q \begin{pmatrix} \delta(q_0^2 + \vec q^2 - 1) \\ \vartheta(q_0)\delta(q_0^2 - \vec q^2 - 1) \end{pmatrix} \\ &= \int d\Omega^{s-1} \int dq_0 \int_0^{\infty} q^{s-1} dq \begin{pmatrix} \delta(q_0^2 + q^2 - 1) \\ \vartheta(q_0)\delta(q_0^2 - q^2 - 1) \end{pmatrix} \end{aligned}$$

The 0-sphere consists of two points

$$\int d\Omega^{s-1} = |\Omega^{s-1}| = \frac{2\pi^{\frac{s}{2}}}{\Gamma(\frac{s}{2})} = 2, 2\pi, 4\pi, 2\pi^2, \frac{8\pi^2}{3}, \ldots, \quad |\Omega^0| = \text{card}\,\{1, -1\}$$

The momentum p-parametrization is square root free for odd position dimension $s = 1, 3, \ldots$, the v-parametrization for all $s = 1, 2, \ldots$ - for the spherical

measures

$$\begin{array}{lll}\int d\Omega^s & = \int d\Omega^{s-1} \frac{1}{2} \int_{-1}^1 \frac{dq_0}{\sqrt{1-q_0^2}^{2-s}} & = \int d\Omega^{s-1} \int_0^1 \frac{q^{s-1}dq}{\sqrt{1-q^2}} \\ & = \int d\Omega^{s-1} \int_0^\infty \frac{2p^{s-1}dp}{\sqrt{1+p^2}^{1+s}} & = \int d\Omega^{s-1} \int_0^\infty v^{s-1}dv \ (\frac{2}{1+v^2})^s \end{array}$$

and the hyperbolic ones

$$\begin{array}{lll}\int d\mathcal{Y}^s & = \int d\Omega^{s-1} \frac{1}{2} \int_1^\infty \frac{dq_0}{\sqrt{q_0^2-1}^{2-s}} & = \int d\Omega^{s-1} \frac{1}{2} \int_0^\infty \frac{q^{s-1}dq}{\sqrt{1+q^2}} \\ & = \int d\Omega^{s-1} \int_0^1 \frac{p^{s-1}dp}{\sqrt{1-p^2}^{1+s}} & = \int d\Omega^{s-1} \frac{1}{2} \int_0^1 v^{s-1}dv \ (\frac{2}{1-v^2})^s \end{array}$$

summarized as follows

$$\begin{array}{llll}\int d\Omega^s & = \int_{\vec{q}^2\le 1} \frac{d^sq}{\sqrt{1-\vec{q}^2}} & = \int \frac{2d^sp}{\sqrt{1+\vec{p}^2}^{1+s}} & = \int d^sv \ (\frac{2}{1+\vec{v}^2})^s \\ \int d\mathcal{Y}^s & = \int \frac{d^sq}{2\sqrt{1+\vec{q}^2}} & = \int_{\vec{p}^2\le 1} \frac{d^sp}{\sqrt{1-\vec{p}^2}^{1+s}} & = \frac{1}{2} \int_{\vec{v}^2\le 1} d^sv \ (\frac{2}{1-\vec{v}^2})^s \end{array}$$

The $\mathbf{SO}(4)$-invariant Ω^3-measure of the momenta $d^3p\ (\frac{2}{Q^2+\vec{p}^2})^2$ has been used above for the bound states in the nonrelativistic Kepler dynamics with Q^2 related to the bound state energy.

Chapter 3

Harmonic Analysis of Interactions

The representations of interactions are different from those of free particles. Spacetime interactions are implemented by the off-shell part in Feynman propagators. They are supported by the causal bicone. The harmonic analysis of functions on the future cone as a homogeneous spacetime model displays the irreducible representations of the acting group. Those representations of the extended Lorentz group $\mathbf{GL}(\mathbb{C}^2)$ have to be used for spacetime interactions. For the harmonic analysis of nonlinear spacetime, free particle fields are not enough. Additional genuine interaction fields with higher order pole energy-momentum measures are necessary[10, 31].

3.1 The Causal Support of Interactions

Feynman propagators, e.g. for a neutral pion field

$$\begin{aligned}\langle\{\mathbf{\Phi},\mathbf{\Phi}\}(x)-\epsilon(x_0)[\mathbf{\Phi},\mathbf{\Phi}](x)\rangle &= \tfrac{i}{\pi}\int\frac{d^4q}{(2\pi)^3}\frac{m}{q^2+io-m^2}e^{iqx}\\ &= m\int\frac{dq_0}{i(2\pi)^2}e^{iq_0x_0}\frac{\vartheta(q_0^2-m^2)\ e^{i|\vec q|r}+\vartheta(m^2-q_0^2)\ e^{-|Q|r}}{r}\\ \text{with}\quad & |\vec q|=\sqrt{q_0^2-m^2},\quad |Q|=\sqrt{m^2-q_0^2}\end{aligned}$$

are the sum of compact and noncompact position representations as seen in the on-shell off-shell decomposition of the energy-momentum measures into free particle and interaction measure

$$\frac{i}{\pi}\frac{1}{q^2+io-m^2}=\delta(q^2-m^2)+\frac{i}{\pi}\frac{1}{q_{\rm P}^2-m^2}$$

The off-shell part $\epsilon(x_0)[\mathbf{\Phi},\mathbf{\Phi}](x)$ with the principal value integration has an explicit spacetime dependence $\epsilon(x_0)\vartheta(x^2)$ which is not implemented by particle fields

$$\frac{i}{\pi}\int\frac{d^4q}{q_{\rm P}^2-m^2}e^{iqx}=-\epsilon(x_0)\int d^4q\ \epsilon(q_0)\delta(q^2-m^2)e^{iqx}$$

Relativistic interaction structures, e.g. Yukawa potentials $\frac{e^{-|Q|r}}{r}$, are supported by the causal spacetime bicone. They arise as Fourier transformed principal values

$$-\int\frac{d^4q}{\pi^3}\frac{1}{q_{\rm P}^2-m^2}e^{iqx}=\ \frac{\partial}{\partial\frac{x^2}{4}}\vartheta(x^2)\mathcal{J}_0(\sqrt{m^2x^2})=\delta(\tfrac{x^2}{4})+\vartheta(x^2)\frac{\partial}{\partial\frac{x^2}{4}}\mathcal{J}_0(\sqrt{m^2x^2})$$

involving the Bessel function $\mathcal{J}_0$.

The advanced energy-momentum pole measures

$$\begin{array}{ll}\frac{1}{i\pi}\frac{1}{(q-io)^2-m^2} & = \epsilon(q_0)\delta(q^2-m^2)+\frac{1}{i\pi}\frac{1}{q_{\rm P}^2-m^2}\\ & \text{with } (q-io)^2=(q_0-io)^2-\vec q^2\end{array}$$

have as Fourier transforms distributions with future support

$$\int \frac{d^4q}{\pi^3}\frac{1}{(q-io)^2-m^2}e^{iqx} = -\vartheta(x_0)\frac{\partial}{\partial\frac{x^2}{4}}\vartheta(x^2)\mathcal{J}_0(\sqrt{m^2x^2})$$

Free particles represent the spacetime translations $\mathbb{R}^4$, interactions the causal bicone $\mathbb{R}^4_\vee \uplus \mathbb{R}^4_\wedge$, future and past.

3.2 Linear and Nonlinear Spacetime

Minkowski translations $\mathbb{R}^4$ (4-dimensional) contain, on the one side, position translations $\mathbb{R}^3$ (3-dimensional) and, on the other side, Cartan translations $\mathbb{R}^2$ (2-dimensional) with the 1-dimensional time and position translations $\mathbb{R}$ resp.

$$\begin{array}{ccccc}\tau\in\mathbb{R} & \longrightarrow & x^0+\sigma_3x^3\in\mathbb{R}^2 & \longrightarrow & x=x^0+\vec x=\begin{pmatrix} x^0+x^3 & x^1-ix^2\\ x^1+ix^2 & x^0-x^3\end{pmatrix}\in\mathbb{R}^4\\ & \nearrow & & \nearrow & \\ z\in\mathbb{R} & \longrightarrow & \vec x\in\mathbb{R}^3 & & \end{array}$$

Cartan and Minkowski translations are parametrized by hermitian (2×2)-matrices, $x^0\cong x^0\mathbf{1}_2$ and $\vec x = x^a\sigma_a$, diagonal for Cartan spacetime. The group $\mathbf{D}(1)=\exp\mathbb{R}$ acts upon the 1-dimensional translations (time and position). For the Cartan translations, it is rearranged to $\mathbf{D}(1)\times\mathbf{D}(1)\cong\mathbf{D}(1)\times\mathbf{SO}_0(1,1)$ with the rotation free orthochronous Lorentz group and then embedded (not unique), together with the position rotations $\mathbf{SO}(3)$, into the $\mathbf{D}(1)$-extended Poincaré group

$$\begin{array}{ccccc}\mathbf{D}(1)\ \vec\times\ \mathbb{R} & \longrightarrow & [\mathbf{D}(1)\times\mathbf{SO}_0(1,1)]\ \vec\times\ \mathbb{R}^2 & \longrightarrow & [\mathbf{D}(1)\times\mathbf{SO}_0(1,3)]\ \vec\times\ \mathbb{R}^4\\ & \nearrow & & \nearrow & \\ \mathbf{D}(1)\ \vec\times\ \mathbb{R} & \longrightarrow & [\mathbf{D}(1)\times\mathbf{SO}(3)]\ \vec\times\ \mathbb{R}^3 & & \end{array}$$

Time future is embedded into Cartan and Minkowski future

$$\begin{array}{llll}\mathbb{R}_\vee\ni\tau_\vee=\vartheta(\tau)\tau & \hookrightarrow & \vartheta(x^0)\vartheta(x^2)(x^0+\sigma_3x^3) & =x_\vee\in\mathbb{R}^2_\vee\\ & \hookrightarrow & \vartheta(x^0)\vartheta(x^2)(x^0+\vec x) & =x_\vee\in\mathbb{R}^4_\vee\end{array}$$

The futures are used as noncompact spaces (open cones without 'skin'), i.e. without the strict presence $x=0$ and without lightlike translations for nontrivial position $s=1,3$

$$x_\vee\in\mathbb{R}^{1+s}_\vee\Rightarrow x^2_\vee>0,\ s=0,1,3$$

Time future is the causal group $\mathbf{D}(1) = \exp \mathbb{R}$

$$\mathbb{R}_\vee \ni \tau_\vee \ = e^t \in \mathbf{D}(1) \cong \mathbf{GL}(\mathbb{C})/\mathbf{U}(1)$$

Cartan future is the direct product of causal group and abelian Lorentz group

$$x_\vee = e^{\psi_0 + \sigma_3 \psi} \in \mathbb{R}^2_\vee \cong \mathbf{D}(1) \times \mathbf{SO}_0(1,1)$$

The action of the full linear group, called extended Lorentz group, on Minkowski translations

$$\mathbf{GL}(\mathbb{C}^2) \times \mathbb{R}^4 \longrightarrow \mathbb{R}^4, \quad g \bullet x = g \circ x \circ g^\star$$

leaves the future invariant. This gives the orbit parametrization with Lie algebra coefficients

$$\begin{array}{rl} x = e^{\psi_0 + \vec{\psi}} = & u(\frac{\vec{\psi}}{\psi}) \circ e^{\psi_0 + \sigma_3 \psi} \circ u(\frac{\vec{\psi}}{\psi})^\star \in \mathbb{R}^4_\vee \cong \mathbf{GL}(\mathbb{C}^2)/\mathbf{U}(2) \\ \text{with} & \left\{ \begin{array}{ll} e^{\psi_0} & = \sqrt{x^2} \in \mathbf{D}(1) \\ e^{\pm\psi} & = \sqrt{\frac{x_0 \pm r}{x_0 \mp r}} \in \mathbf{SO}_0(1,1), \quad |\vec{\psi}| = \psi, \quad \frac{\vec{\psi}}{\psi} = \frac{\vec{x}}{r} \end{array} \right. \end{array}$$

There are two rotation degrees of freedom for the 2-sphere $\Omega^2 \cong \mathbf{SU}(2)/\mathbf{SO}(2)$

$$u(\tfrac{\vec{\psi}}{\psi}) = \begin{pmatrix} \cos\frac{\theta}{2} & ie^{-i\varphi}\sin\frac{\theta}{2} \\ ie^{i\varphi}\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} = \frac{1}{\sqrt{2r(r+x_3)}} \begin{pmatrix} r + x_3 & -ix_1 - x_2 \\ ix_1 - x_2 & r + x_3 \end{pmatrix} \in \mathbf{SU}(2)/\mathbf{SO}(2)$$

The futures are irreducible orbits of $\mathbf{D}(1) \times \mathbf{SO}_0(1,s)$. 1-dimensional and 4-dimensional future are the first two entries in the symmetric space chain $\mathbf{GL}(\mathbb{C}^n)/\mathbf{U}(n)$, $n = 1, 2, \ldots$, which are the manifolds of the unitary groups in the general linear group, canonically parametrized in the polar decomposition $g = u \circ |g|$ with the real n^2-dimensional ordered absolute values $x_\vee = |g| = \sqrt{g^\star \circ g} \in \mathbb{R}^{n^2}_\vee$ of the general linear group. They are the positive cone of the ordered C*-algebras with the complex $n \times n$-matrices.

The tangent space of future at each point is a full translation space

$$\mathbb{R}^{1+s} \cong \left\{ \begin{array}{ll} \log \mathbf{D}(1), & s = 0 \\ \log \mathbf{D}(1) \ \oplus \ \log \mathbf{SO}_0(1,1), & s = 1 \\ \log \mathbf{GL}(\mathbb{C}^2)/\mathbf{U}(2), & s = 3 \end{array} \right.$$

The Lie algebra coefficients $(\psi_0, \vec{\psi}) \in \mathbb{R} \times \mathbb{R}^3$ are related to tangent time and position as follows

$$x_\vee = x_0 + \vec{x} = e^{\psi_0}(\cosh\psi + \tfrac{\vec{\psi}}{\psi}\sinh\psi) = 1 + \psi_0 + \vec{\psi} + \ldots$$

Minkowski future contains many familiar homogeneous subspaces in the manifold decomposition

$$\mathbf{D}(2) \cong \mathbf{D}(1) \times \mathbf{SL}(\mathbb{C}^2)/\mathbf{SU}(2)$$

1-dimensional future $\mathbf{D}(1)$ is multiplied with the 3-dimensional Lobatchevski space, i.e. with the 3-hyperboloid

$$\begin{aligned} \tfrac{x}{\sqrt{x^2}} = e^{\vec{\psi}} \in \mathcal{Y}^3 & \cong \mathbf{SL}(\mathbb{C}^2)/\mathbf{SU}(2) \cong \mathbf{SO}_0(1,3)/\mathbf{SO}(3) \\ & \cong \mathcal{Y}^2 \times \Omega^1 \cong \mathcal{Y}^1 \times \Omega^2 \end{aligned}$$

which contains 2-dimensional non-Euclidean planes (2-hyperboloids)

$$\mathcal{Y}^2 \cong \mathbf{SU}(1,1)/\mathbf{SO}(2) \cong \mathbf{SO}_0(1,2)/\mathbf{SO}(2)$$

and 1-dimensional hyperboloids (abelian Lorentz groups) $\mathcal{Y}^1 \cong \mathbf{SO}_0(1,1)$. The 2-sphere and 2-hyperboloid are $\mathbf{SO}(2)$-orbits of Ω^1 and $\mathcal{Y}^1$ for the axial rotations $o(\varphi)$

$$\begin{aligned} & \mathcal{Y}^2 \cong \mathcal{Y}^1 \times \Omega^1, \quad \Omega^2 \cong \Omega^1 \times \Omega^1 \\ & \begin{pmatrix} \cosh\frac{\psi}{2} & e^{-i\varphi}\sinh\frac{\psi}{2} \\ e^{i\varphi}\sinh\frac{\psi}{2} & \cosh\frac{\psi}{2} \end{pmatrix} = o(\varphi) \circ e^{\sigma_3 \frac{\psi}{2}} \circ o^\star(\varphi) \\ & o(\varphi) = \tfrac{1}{\sqrt{2}}\begin{pmatrix} 1 & e^{-i\varphi} \\ -e^{i\varphi} & 1 \end{pmatrix} \in \mathbf{SU}(2), \quad \begin{cases} e^{\sigma_3 \frac{\psi}{2}} & \in \mathbf{SO}_0(1,1) \\ e^{i\sigma_3 \frac{\chi}{2}} & \in \mathbf{SO}(2) \text{ for } \psi = i\chi \end{cases} \end{aligned}$$

Here, and in general, the decompositions and isomorphies are with respect to the manifold structure, not with respect to the symmetric space structure, e.g., Ω^2 is symmetric with respect to $\mathbf{SO}(3)$, Ω^1 only with respect to $\mathbf{SO}(2)$.

3.3 Harmonic Analysis of Spacetime

Harmonic analysis of a symmetric space G/H with Lie real groups $G \supseteq H$ analyzes the square integrable functions $f \in L^2(G/H)$ with respect to a decomposition into irreducible representation spaces of G with the related eigenvalues and invariants. Hilbert spaces with irreducible faithful representations of noncompact groups are infinite dimensional.

Harmonic analysis of the tangent groups for linear spacetime

$$\begin{array}{ccccc} \mathbb{R} & \longrightarrow & \mathbf{SO}_0(1,1) \,\vec{\times}\, \mathbb{R}^2 & \longrightarrow & \mathbf{SO}_0(1,3) \,\vec{\times}\, \mathbb{R}^4 \\ & \nearrow & & \nearrow & \\ \mathbb{R} & \longrightarrow & \mathbf{SO}(3) \,\vec{\times}\, \mathbb{R}^3 & & \end{array}$$

involves the irreducible Hilbert representations for free scattering states and free particles (Wigner), i.e. the square integrable functions on affine groups (tangent groups) $L^2(\mathbf{SO}(s) \,\vec{\times}\, \mathbb{R}^s)$ and $L^2(\mathbf{SO}_0(1,s) \,\vec{\times}\, \mathbb{R}^{1+s})$, $s \geq 1$.

Harmonic analysis of nonlinear spacetime $\mathbf{D}(2) \cong \mathbf{D}(1) \times \mathcal{Y}^3$

$$\begin{array}{ccccc} \mathbf{D}(1) & \longrightarrow & \mathbf{D}(1) \times \mathcal{Y}^1 & \longrightarrow & \mathbf{D}(2) \\ & \nearrow & & \nearrow & \\ \mathcal{Y}^1 & \longrightarrow & \mathcal{Y}^3 & & \end{array}$$

embeds the harmonic analysis of the following flat, spherical and hyperbolic symmetric subspaces

$\dim_{\mathbb{R}}$	flat	spherical	hyperbolic
1	$\mathbf{D}(1) \cong \mathbb{R}$	$\Omega^1 \cong \mathbf{SO}(2)$	$\mathcal{Y}^1 \cong \mathbf{SO}_0(1,1)$
2		$\Omega^2 \cong \mathbf{SO}(3)/\mathbf{SO}(2)$	$\mathcal{Y}^2 \cong \mathbf{SO}_0(1,2)/\mathbf{SO}(2)$
3			$\mathcal{Y}^3 \cong \mathbf{SO}_0(1,3)/\mathbf{SO}(3)$

The eigenvalues in a group G representation are linear forms $(\log G)^T$ of the Lie algebra $\log G$, for spacetime called angular momenta and spin in the case of spherical spaces and energies and momenta for flat and hyperbolic spaces. The Lie algebra $\log G$ is acted upon with the adjoint representation of the group G, its forms $\log G^T$ with the coadjoint (dual) one. Invariants are multilinear Lie algebra forms, e.g. the bilinear Killing form.

The tangent spaces of G/H, all isomorphic to each other, are isomorphic to the corresponding Lie algebra classes and denoted by $\log G/H = \log G/\log H$ with $\dim_{\mathbf{R}} \log G/H = \dim_{\mathbf{R}} G - \dim_{\mathbf{R}} H$. They inherit the adjoint action of the group G, equally its linear forms.

Harmonic analysis of all the homogeneous spaces above is well known [28, 15, 7, 26, 12, 3] and will be shortly repeated.

3.3.1 The Abelian Groups

For the axial rotations $\mathbf{SO}(2) \cong \Omega^1$ with rank 1 Lie algebra $\log \mathbf{SO}(2) \cong \mathbb{R}$ there are the Fourier series with the winding numbers $l \in \mathbb{Z}$ as invariants. They characterize the irreducible representations of the circle as integer powers of the defining representation

$$\begin{array}{rcl} e^{\pm i\chi} \cong \begin{pmatrix} \cos\chi \\ i\sin\chi \end{pmatrix} \in \Omega^1 & \leftrightarrow & l \in \mathbb{Z} \\ \text{irreducible representations: } e^{\pm i\chi} & \longmapsto & (e^{i\chi})^l \\ f(\chi) = \sum_{l=-\infty}^{\infty} e^{il\chi} \tilde f(l) & \leftrightarrow & \tilde f(l) = \int_{-\pi}^{\pi} \frac{d\chi}{2\pi}\, e^{-il\chi} f(\chi) \end{array}$$

Harmonic (Fourier) analysis of the causal group $\mathbf{D}(1) \cong \mathbb{R}$ has linear invariants (energies) in a continuous spectrum $E \in [\log \mathbf{D}(1)]^T \cong \mathbb{R}$

$$\begin{array}{rcl} e^t \in \mathbf{D}(1) & \leftrightarrow & E \in \mathbb{R} \\ \text{irreducible representations: } e^t & \longmapsto & (e^t)^{iE} = e^{itE} \\ f(t) = \int \frac{dE}{2\pi} e^{iEt} \tilde f(E) & \leftrightarrow & \tilde f(E) = \int dt\, e^{-iEt} f(t) \end{array}$$

The irreducible Hilbert representations arise by the transition to an imaginary power of the defining nonunitary representation $e^t \longmapsto (e^t)^i$ and, then, by a continuous real power $e^{it} \longmapsto (e^{it})^E$ of this unitary representation.

Harmonic analysis of the abelian Lorentz group $\mathbf{SO}_0(1,1) \cong \mathcal{Y}^1 \times \Omega^0$ (1-dimensional position with $\psi = z = x_3$) coincides with the analysis of flat space $\mathbf{D}(1)$ which will be rewritten in an adapted parametrization

$$\begin{array}{rcl} e^{\pm\psi} \cong \begin{pmatrix} \cosh\psi \\ \sinh\psi \end{pmatrix} \in \mathcal{Y}^1 & \leftrightarrow & P = \pm|P| \in \mathbb{R} \\ \text{irreducible representations: } e^{\pm\psi} & \longmapsto & (e^{\pm\psi})^{i|P|} \\ f(\psi) = \int_0^{\infty} \frac{dP}{2\pi} \sum_{\epsilon=-1}^{1} e^{i\epsilon P\psi} \tilde f(|P|,\epsilon) & \leftrightarrow & \tilde f(|P|,\epsilon) = \int d\psi\, e^{-i\epsilon|P|} f(\psi) \end{array}$$

The Plancherel measure of the irreducible Hilbert representations of $\mathbf{SO}(2)$ and $\mathbf{SO}_0(1,1)$ are the counting and Lebesque measure

$$\sum_{l\in\mathbb{Z}} = \sum_{L=0}^{\infty}\sum_{\epsilon=-1}^{1} \text{ and } \int \frac{dP}{2\pi} = \int_0^\infty \frac{dP}{2\pi}\sum_{\epsilon=-1}^{1}$$

3.3.2 Spheres

Compact structures are analyzed with the Peter-Weyl theorem[24]. The analysis of the 2-sphere $\Omega^2 \cong \mathbf{SO}(3)/\mathbf{SO}(2)$ uses the representation of the rotations with rank 1 Lie algebra $\log \mathbf{SO}(3) \cong \mathbb{R}^3$. The eigenvalues (magnetic quantum numbers) are a discrete subset of a 1-dimensional tangent form subspace $L_3 \in \mathbb{Z} \subset \mathbb{R} \subset [\log \mathbf{SO}(3)]^T \cong \mathbb{R}^3$. The characterizing Casimir invariant is the bilinear Killing form with its values $L(L+1)$ determining the angular momenta

$$\begin{array}{rcl}
\frac{\vec{x}}{r} = \begin{pmatrix}\cos\theta\\ i\sin\theta e^{\pm i\varphi}\end{pmatrix} \in \Omega^2 & \leftrightarrow & (L, L_3) \in \mathbb{N}\times\mathbb{Z}\\
\text{irreducible representations: } \frac{\vec{x}}{r} & \longmapsto & (\frac{\vec{x}}{r})^L \text{ (traceless)}\\
f(\frac{\vec{x}}{r}) = \sum_{L=0}^{\infty}\sum_{L_3=-L}^{L} (\frac{\vec{x}}{r})^L_{L_3}\tilde{f}(L)^{L_3} & \leftrightarrow & \tilde{f}(L)^{L_3} = \int \frac{d\Omega^2}{4\pi}\overline{(\frac{\vec{x}}{r})^L_{L_3}} f(\frac{\vec{x}}{r})\\
\text{Plancherel measure:} & & \sum_{L=0}^{\infty}
\end{array}$$

The spherical harmonics as basis of the irreducible $\mathbf{SO}(3)$-representation spaces are the natural powers of the directions in $\mathbb{R}^3$, i.e. the possible $\mathbf{SO}(2)$'s in $\mathbf{SO}(3)$, i.e. traceless monomials of the defining representation $\sqrt{\frac{4\pi}{1+2L}}\mathrm{Y}^L(\frac{\vec{x}}{r}) \cong (\frac{\vec{x}}{r})^L$.

This structure is characteristic: E.g., the 3-sphere $\Omega^3 \cong \mathbf{SO}(4)/\mathbf{SO}(3)$ has as defining representations unit vectors $q \in \mathbb{R}^4$, $q^2 = 1$ whereof the Kepler representations $\mathrm{Y}^{(L,L)}(q)$ in the harmonic analysis are natural powers.

3.3.3 Tangent Groups

The irreducible Hilbert representations of the Euclidean group in the plane $\mathbf{SO}(2) \vec{\times} \mathbb{R}^2$ are - for nontrivial translation invariant - infinite dimensional. The matrix elements

$$\begin{array}{rl}
m^2 > 0: & \mathbb{R}^2 \ni \vec{x} \longmapsto \int d^2q\, \delta(\vec{q}^2 - m^2)e^{-i\vec{q}\vec{x}} = \pi\mathcal{J}_0(mr)\\
& \mathbf{SO}(2)\text{-nontrivial with } (\vec{q})^L \sim (i\frac{\partial}{\partial\vec{x}})^L, \quad L = 0, 1, \dots\\
& \qquad \frac{\partial}{\partial\vec{x}} = \frac{\vec{x}}{r}\frac{\partial}{\partial r} = \frac{\vec{x}}{2}\frac{\partial}{\partial\frac{r^2}{4}}
\end{array}$$

involve integer index Bessel functions

$$\begin{array}{rl}
\mathbb{R} \ni \xi \longmapsto \mathcal{E}_L(\frac{\xi^2}{4}) & = \frac{\mathcal{J}_L(\xi)}{(\frac{\xi}{2})^L} = \sum_{n=0}^{\infty}\frac{(-\frac{\xi^2}{4})^n}{(L+n)!n!} = \left(-\frac{\partial}{\partial\frac{\xi^2}{4}}\right)^L \mathcal{E}_0(\frac{\xi^2}{4})\\
\mathcal{E}_0(\frac{\xi^2}{4}) & = \mathcal{J}_0(\xi), \quad (1+L)\mathcal{E}_{1+L}(\frac{\xi^2}{4}) = \mathcal{E}_L(\frac{\xi^2}{4}) + \frac{\xi^2}{4}\mathcal{E}_{2+L}(\frac{\xi^2}{4})
\end{array}$$

The irreducible Hilbert representations of the Euclidean group in 3-dimensional position $\mathbf{SO}(3) \ \vec{\times}\ \mathbb{R}^3$ for nontrivial translation invariant are induced from momentum fixgroup $\mathbf{SO}(2)$ representations with the matrix elements

$$\begin{array}{rl} m^2 > 0: & \mathbb{R}^3 \ni \vec x \longmapsto \int d^3q \ \frac{1}{2|m|} \ \delta(\vec q^2 - m^2) e^{-i\vec q\vec x} = \pi j_0(mr) = \pi \frac{\sin mr}{mr} \\ & \mathbf{SO}(3)\text{-nontrivial with } (\vec q)^L \sim (i\frac{\partial}{\partial \vec x})^L, \quad L = 2J = 0, 1, \dots \\ & \frac{\partial}{\partial \vec x} = \frac{\vec x}{r}\frac{\partial}{\partial r} = \frac{\vec x}{2}\frac{\partial}{\partial \frac{r^2}{4}} \end{array}$$

They are products $j_L(mr)\mathrm{P}^L(\cos\theta)$ of harmonic polynomials in one direction (Legendre polynomials $\mathrm{P}^L \sim \mathrm{Y}^L_0$) with matching spherical (halfinteger index) Bessel functions

$$\mathbb{R} \ni \xi \longmapsto \frac{2}{\sqrt\pi}\frac{j_L(\xi)}{(\frac{\xi}{2})^L} = \frac{\mathcal{J}_{L+\frac12}(\xi)}{(\frac{\xi}{2})^{L+\frac12}} = \sum_{n=0}^\infty \frac{(-\frac{\xi^2}{4})^n}{\Gamma(\frac32+L+n)n!}$$

The irreducible Hilbert representations of the rotation free Poincaré group $\mathbf{SO}_0(1,1) \ \vec{\times}\ \mathbb{R}^2$ with Cartan translations are - for nontrivial translation invariant - characterized by the matrix elements

$$\begin{array}{rl} m^2 \neq 0: \ \mathbb{R}^2 \ni x \longmapsto & \int d^2q \ \delta(q^2 - m^2) e^{iqx} = -\pi\mathcal{N}_0(\sqrt{m^2x^2}) \\ & = -\pi\vartheta(x^2\mathcal{N}_0(\sqrt{m^2x^2}) + 2vth(-x^2)\mathcal{K}_0(\sqrt{-m^2x^2}) \\ & \mathbf{SO}_0(1,1)\text{-nontrivial with } (q)^L \sim (-i\frac{\partial}{\partial x})^L, \quad L = 0, 1, \dots \\ & \frac{\partial}{\partial x} = \frac{x}{2}\frac{\partial}{\partial \frac{x^2}{4}} \end{array}$$

The order 0 Neumann function $\mathcal{N}_0$ for real argument (timelike) is the Macdonald function $\mathcal{K}_0$ for imaginary argument (spacelike)

$$\begin{array}{rl} \mathbb{R} \ni \xi \longmapsto \pi\mathcal{N}_0(\xi) & = \sum_{n=0}^\infty \frac{(-\frac{\xi^2}{4})^n}{(n!)^2}[\log\frac{|\xi^2|}{4} + 2\gamma_0 - 2\varphi(n)] = -2\mathcal{K}_0(-i\xi) \\ \varphi(0) & = 0, \quad \varphi(n) = 1 + \frac12 + \dots + \frac1n, \quad n = 1, 2, \dots \\ \gamma_0 & = -\Gamma'(1) = \lim_{n\to\infty}[\varphi(n) - \log n] = 0.5772\dots \end{array}$$

3-dimensional Fourier transformations are related to 1-dimensional ones by a radial derivative which produces the Kepler factor $\frac1r$

$$\int d^3q \ \mu(\vec q^2) e^{-i\vec q\vec x} = -\frac{\partial}{\partial\frac{r^2}{4\pi}} \int dq \ \mu(q^2) e^{-iqr}$$

as seen for the Euclidean group above with the 2-sphere spread of $\mathbf{SO}(2)$-representation matrix elements, e.g. $2\frac{\sin\xi}{\xi} = -\frac{\partial}{\partial\frac{\xi^2}{4}}\cos\xi$. Equally, the Fourier transformations in Minkowski spacetime are obtainable from Cartan spacetime by an invariant derivation (2-sphere spread)

$$\int d^4q \ \binom{1}{\epsilon(q_0)\vartheta(q^2)} \mu(q^2) e^{iqx} = \frac{\partial}{\partial\frac{x^2}{4\pi}} \int d^2q \ \binom{1}{\epsilon(q_0)\vartheta(q^2)} \mu(q^2) e^{iqx}\Big|_{x=(x_0,r)}$$

Therewith, the irreducible Hilbert representations of the Poincaré group $\mathbf{SO}_0(1,3) \ \vec{\times}\ \mathbb{R}^4$ for Minkowski translations contain - for causal translation

invariant $m^2 \geq 0$ with the representation inducing fixgroups $\mathbf{SU}(2)$ and $\mathbf{SO}(2)$ - the Wigner classified particle representations with matrix elements

$$\begin{array}{ll} m^2 > 0: & \mathbb{R}^4 \ni x \longmapsto \int d^4q \ \frac{1}{m^2} \ \delta(q^2 - m^2) e^{iqx} = -\frac{\partial}{\partial \frac{m^2 x^2}{4\pi}} \pi \mathcal{N}_0(\sqrt{m^2 x^2}) \\ & \mathbf{SO}_0(1,3)\text{-nontrivial with } (q)^L \sim (-i\frac{\partial}{\partial x})^L, \quad L = 2J = 0, 1, \ldots \\ & \qquad \frac{\partial}{\partial x} = \frac{x}{2} \frac{\partial}{\partial \frac{x^2}{4}} \end{array}$$

For nontrivial spin invariant J there arise the corresponding Lorentz compatible energy-momentum powers $(q)^{2J}$.

3.3.4 Hyperboloids

The harmonic analysis of the abelian group $\mathbf{SO}_0(1,1) \cong \mathcal{Y}^1$ (1-dimensional position) above can be formulated in Cartan spacetime $\mathbb{R}^2$ which embeds both the hyperboloid $\mathcal{Y}^1$ and all its tangent spaces $\log \mathbf{SO}_0(1,1) \cong \mathbb{R}$. With the backward-forward momenta $\epsilon|P|$, $\epsilon = \pm 1 \in \Omega^0$, as linear forms thereon, the defining $\mathbf{SO}_0(1,1)$-representation can be written as Lorentz product $Py = P_0 y_0 - P_1 y_1$ with a lightlike energy-momentum P, i.e. $P^2 = 0$, normalized with the momentum $|P|$

$$\mathcal{Y}^1 \ni y = \begin{pmatrix} y_0 \\ y_1 \end{pmatrix} = \begin{pmatrix} \cosh\psi \\ \sinh\psi \end{pmatrix}, \quad P = |P| \begin{pmatrix} 1 \\ \epsilon \end{pmatrix} \in \mathbb{R}_+ \times \Omega^0, \ \frac{Py}{|P|} = \{e^{\epsilon\psi}\}$$

The analogue notation in Euclidean $\mathbb{R}^2$ for the defining representation of $\mathbf{SO}(2) \cong \Omega^1$ reads with the Euclidean product $x^2 = x_1^2 + x_2^2$

$$\Omega^1 \ni x = \begin{pmatrix} \cos\chi \\ i\sin\chi \end{pmatrix}, \quad l = L \begin{pmatrix} 1 \\ \epsilon \end{pmatrix} \in \mathbb{N} \times \Omega^0, \quad \frac{lx}{L} = \{e^{i\epsilon\chi}\}$$

There are other parametrizations: The $\mathbf{SO}_0(1,1)$-exponentials $e^{\pm\psi}$ are the lightcone coordinates $y_0 \pm y_1$ in the tangent parametrization and $\frac{1 \pm X}{1 \mp X}$ in the Poincaré parametrization

$$\begin{array}{rcl} y = \frac{1}{\sqrt{1-x^2}} \begin{pmatrix} 1 \\ x \end{pmatrix} = \frac{2}{1-X^2} \begin{pmatrix} \frac{1+X^2}{2} \\ X \end{pmatrix} \in \mathcal{Y}^1 & \leftrightarrow & P = |P| \begin{pmatrix} 1 \\ \pm 1 \end{pmatrix} \in \mathbb{R}_+ \times \Omega^0 \\ e^{\pm\psi} & = & y_0 \pm y_1 = \sqrt{\frac{1 \pm x}{1 \mp x}} = \frac{1 \pm X}{1 \mp X} \\ f(y) = \int \frac{dP}{2\pi} \left(\frac{Py}{|P|}\right)^{i|P|} \tilde{f}(P) & \leftrightarrow & \tilde{f}(P) = \int d\mathcal{Y}^1(y) \ \left(\frac{Py}{|P|}\right)^{-i|P|} f(y) \end{array}$$

The $\mathcal{Y}^1$-measure has the corresponding parametrizations

$$\int d\mathcal{Y}^1 = \int_0^\infty d\psi \int d\Omega^0 = \int \frac{dy_1}{\sqrt{1+y_1^2}} = \int_{-1}^1 \frac{dx}{1-x^2} = \int_{-1}^1 \frac{dX}{1-X^2}$$

The Lorentz compatible formalism with the powers of the defining $\mathbf{SO}_0(1,1)$-representations $(\frac{Py}{|P|})^{i|P|} = (e^{\epsilon\psi})^{i|P|}$ as irreducible representations is generalizable to higher dimensions $\mathbb{R}^{1+s}$ with $\mathbf{SO}_0(1,s)$-invariant products: The hyperboloid $\mathcal{Y}^s$ (nonlinear s-dimensional position) embeds flat hyperboloids $\mathcal{Y}^1$ (1-dimensional positions), related to each other by rotations $\mathbf{SO}(s)$. With $\mathcal{Y}^s$ and its tangent spaces $\mathbb{R}^s$ embedded in $\mathbb{R}^{1+s}$, the harmonic analysis of the

$\mathcal{Y}^s$-functions use the tangent space forms $\vec{P} \in \mathbb{R}^s$ (momenta). The future lightcone V^s in $\mathbb{R}^{1+s}$

$$\mathrm{V}^s \cong \mathbf{SO}_0(1,s)/[\mathbf{SO}(s-1) \vec{\times} \mathbb{R}^{s-1}] \cong \mathbb{R}_+ \times \Omega^{s-1}$$

is isomorphic to the $\mathcal{Y}^s$-tangent space. The momenta $\vec{P} \in (\log \mathcal{Y}^s)^T \cong \mathrm{V}^s$ can be written as lightlike energy-momenta $\mathrm{V}^s = \{P \in \mathbb{R}^{1+s} \mid P^2 = 0\}$

$$\begin{array}{ccc} \mathbb{R}^{1+s} \supset \mathcal{Y}^s \ni y = \begin{pmatrix} y_0 \\ \vec{y} \end{pmatrix} = \begin{pmatrix} \cosh\psi \\ \frac{\vec{y}}{r}\sinh\psi \end{pmatrix} & \leftrightarrow & \mathbb{R}^{1+s} \supset \mathrm{V}^s \ni P = |\vec{P}| \begin{pmatrix} 1 \\ \frac{\vec{P}}{|\vec{P}|} \end{pmatrix} \\ y^2 = 1, & & P^2 = 0 \end{array}$$

The normalized rotation $\mathbf{SO}(s)$-invariant product contains the defining representation of the hyperboloids $\mathbf{SO}_0(1,1)$ in $\mathcal{Y}^s$, indexed with an Ω^1-orientation angle α

$$\begin{aligned} \frac{Py}{|\vec{P}|} &= \cosh\psi + \sinh\psi\cos\alpha = e^\psi \cos^2\frac{\alpha}{2} + e^{-\psi}\sin^2\frac{\alpha}{2} \\ &= |\cosh\frac{\psi}{2} + e^{-i\alpha}\sinh\frac{\psi}{2}|^2, \quad \cos\alpha = \frac{\vec{P}\vec{y}}{|\vec{P}|r} \end{aligned}$$

The irreducible representations for the noncompact abelian subgroups $\mathcal{Y}^1(\alpha) \cong \mathbf{SO}_0(1,1)$ have an imaginary power

$$\text{for all } \mathcal{Y}^1(\alpha): \quad \frac{Py}{|\vec{P}|} \longmapsto (e^\psi \cos^2\frac{\alpha}{2} + e^{-\psi}\sin^2\frac{\alpha}{2})^{i|\vec{P}|} = (\frac{Py}{|\vec{P}|})^{i|\vec{P}|}$$

Therewith, the position functions $L^2(\mathcal{Y}^s)$ have the harmonic analysis

$$\begin{aligned} f(\vec{\psi}) &= \int d\check{\mathcal{Y}}^s(\vec{P}) \frac{(e^\psi\cos^2\frac{\alpha}{2}+e^{-\psi}\sin^2\frac{\alpha}{2})^{i|\vec{P}|}}{(e^\psi\cos^2\frac{\alpha}{2}+e^{-\psi}\sin^2\frac{\alpha}{2})^{\frac{s-1}{2}}}\tilde{f}(\vec{P}) \\ \leftrightarrow \tilde{f}(\vec{P}) &= \int d\mathcal{Y}^s(\vec{\psi}) \frac{(e^\psi\cos^2\frac{\alpha}{2}+e^{-\psi}\sin^2\frac{\alpha}{2})^{-i|\vec{P}|}}{(e^\psi\cos^2\frac{\alpha}{2}+e^{-\psi}\sin^2\frac{\alpha}{2})^{\frac{s-1}{2}}} f(\vec{\psi}) \end{aligned}$$

The irreducible representations are normalized with a length factor for the corresponding compact sphere $\mathcal{Y}^s \cong \mathcal{Y}^1(\alpha) \times \Omega^{s-1}(\alpha)$

$$|\cosh\frac{\psi}{2} + e^{-i\alpha}\sinh\frac{\psi}{2}|^{s-1} = (\frac{Py}{|\vec{P}|})^{\frac{s-1}{2}}$$

The Plancherel measure[26] Π^s for the quadratic invariants $\vec{P}^2$ of the arising irreducible $\mathbf{SO}_0(1,s)$-representations reads

$$\begin{aligned} \int d\check{\mathcal{Y}}^s(\vec{P}) &= \frac{1}{(2\pi)^s}\int_0^\infty \Pi^s(\vec{P}^2)dP \int d\Omega^{s-1} \\ \Pi^s(m^2) &= \left|\frac{\Gamma(im+\frac{s-1}{2})}{\Gamma(im)}\right|^2 = \begin{cases} m\tanh\pi m, & s=2 \\ m^2, & s=3 \end{cases} \end{aligned}$$

E.g., the harmonic analysis of the $\mathcal{Y}^2$-functions (2-dimensional hyperbolic position, non-Euclidean plane) employs the principal series representations[20]

of the group $\mathbf{SL}(\mathbb{R}^2) \sim \mathbf{SO}_0(1,2)$. The Plancherel with Ω^1-measure is Lebesque d^2P with a hyperbolic factor

$$\begin{array}{rcl} y \in \mathcal{Y}^2 \cong \mathcal{Y}^1 \times \Omega^1 & \leftrightarrow & P = |\vec{P}|\begin{pmatrix} 1 \\ \cos\Phi \\ \sin\Phi \end{pmatrix} \in \mathrm{V}^2 \cong \mathbb{R}_+ \times \Omega^1 \\ y = \begin{pmatrix} y_0 \\ y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \cosh\psi \\ \sinh\psi\cos\varphi \\ \sinh\psi\sin\varphi \end{pmatrix} & = & \frac{1}{\sqrt{1-x^2}}\begin{pmatrix} 1 \\ \vec{x} \end{pmatrix} = \frac{2}{1-X^2}\begin{pmatrix} \frac{1+X^2}{2} \\ \vec{X} \end{pmatrix} \\ \frac{Py}{|\vec{P}|} = \cosh\psi + \sinh\psi\cos(\varphi-\Phi) = \frac{1+X^2+2X\cos(\varphi-\Phi)}{1-X^2} \\ \int d\check{\mathcal{Y}}^2(\vec{P}) & = & \int \frac{d^2P}{(2\pi)^2}\ \tanh\pi|\vec{P}| \end{array}$$

Also the harmonic analysis of the functions on the hyperboloid (nonlinear 3-dimensional position) $\mathcal{Y}^3 \cong \mathbf{SL}(\mathbb{C}^2)/\mathbf{SU}(2)$ employs the principal series representations[16] of the Lorentz group $\mathbf{SL}(\mathbb{C}^2) \sim \mathbf{SO}_0(1,3)$. Only for $s=1$ and for $s=3$ the Plancherel with Ω^{s-1}-measure is Lebesque d^sP

$$\begin{array}{rcl} y = \begin{pmatrix} y_0 \\ \vec{y} \end{pmatrix} \in \mathcal{Y}^3 \cong \mathcal{Y}^1 \times \Omega^2 & \leftrightarrow & P = |\vec{P}|\begin{pmatrix} 1 \\ \frac{\vec{P}}{|\vec{P}|} \end{pmatrix} \in \mathrm{V}^3 \cong \mathbb{R}_+ \times \Omega^2 \\ \int d\check{\mathcal{Y}}^3(\vec{P}) & = & \int \frac{d^3P}{(2\pi)^3} \end{array}$$

There is the $L^2(\mathcal{Y}^s)$-analogue treatment[25] for the harmonic analysis of the square integrable functions $L^2(\Omega^s)$ on the sphere in Euclidean $\mathbb{R}^{1+s}$. With the hyperboloid-sphere transition $\psi \leftrightarrow i\chi$

$$\mathbb{R}^{1+s} \supset \Omega^s \ni x = \begin{pmatrix} x_0 \\ i\vec{x} \end{pmatrix} = \begin{pmatrix} \cos\chi \\ i\frac{\vec{x}}{r}\sin\chi \end{pmatrix} \leftrightarrow \mathbb{R}^{1+s} \supset l = L\begin{pmatrix} 1 \\ \frac{\vec{l}}{L} \end{pmatrix},\quad L = |\vec{l}|$$

one uses the representations of the abelian subgroups $\Omega^1(\alpha) \cong \mathbf{SO}(2)$

$$\begin{array}{rl} \text{for all } \Omega^1(\alpha): \quad \frac{lx}{L} & = \cos\chi + i\sin\alpha\sin\chi = e^{i\chi}\cos^2\frac{\alpha}{2} + e^{-i\chi}\sin^2\frac{\alpha}{2} \\ & \longmapsto (\frac{lx}{L})^L,\quad \cos\alpha = \frac{\vec{l}\vec{x}}{Lr} \end{array}$$

3.4 Residual Representations of Nonlinear Spacetime

The defining representations of the abelian groups above can be written as residues[1] from Fourier transformed energy-momentum measures[41] where the poles are at the value of the invariant - real for compact and imaginary for noncompact transformations

$$\begin{array}{rl} \mathbf{U}(1) \cong \Omega^1: \quad e^{i\chi} = & \frac{1}{2i\pi}\oint \frac{dq}{q-1}e^{iq\chi} = \int dq\ \delta(q-1)e^{iq\chi} \\ \mathbf{D}(1) \cong \mathcal{Y}^1: \quad e^{-|\psi|} = & \frac{1}{\pi}\int \frac{dq}{q^2+1}e^{-iq\psi} \text{ with } \int d\Omega^1 = \int \frac{2dq}{q^2+1} = 2\pi \end{array}$$

This defines basic representation measures - the Dirac measure for the sphere Ω^1 and the spherical measure for the hyperboloid $\mathcal{Y}^1$

$$\begin{array}{rl} \Omega^1: \quad e^{i\chi} & \leftrightarrow dq\ \delta(q-1) \\ \mathcal{Y}^1: \quad e^{-|\psi|} & \leftrightarrow \frac{1}{\pi}\frac{dq}{q^2+1} = \frac{d\Omega^1}{|\Omega^1|} \end{array}$$

The measures are normalizable with a residue 1 for the invariant leading to the unit for the neutral group element.

A rescaling changes the representation invariant

$$e^{il\chi} = \frac{1}{2i\pi}\oint \frac{dq}{q-l}e^{iq\chi}, \quad e^{-|Q\psi|} = \frac{1}{\pi}\int \frac{dq|Q|}{q^2+Q^2}e^{-iq\psi}$$

With additional Ω^2-degrees of freedom the rotation group is represented with a derived Dirac measure

$$\mathbf{SU}(2) \cong \Omega^1 \times \Omega^2 : \quad \begin{cases} e^{i\vec{x}} \longmapsto & \frac{1}{\pi}\int d^3q\ \delta'(1-\vec{q}^2)e^{-i\vec{q}\vec{x}}\begin{pmatrix} 1 \\ \vec{q} \end{pmatrix} = \begin{pmatrix} \cos|\vec{x}| \\ -i\frac{\vec{x}}{|\vec{x}|}\ \sin|\vec{x}| \end{pmatrix} \\ \quad \leftrightarrow & \frac{d^3q}{\pi}\delta'(1-\vec{q}^2) \end{cases}$$

and the defining Hilbert representations of Lobachevski space with the scalar dipole measure of the 3-sphere and the vector tripole measure

$$\mathcal{Y}^3 \cong \mathcal{Y}^1 \times \Omega^2 : \quad \begin{cases} e^{\vec{\psi}} \longmapsto & \frac{1}{\pi^2}\int \frac{d^3q}{(\vec{q}^2+1)^2}e^{-i\vec{q}\vec{\psi}}\begin{pmatrix} 1 \\ \frac{4i\vec{q}}{\vec{q}^2+1} \end{pmatrix} = \begin{pmatrix} 1 \\ \vec{\psi} \end{pmatrix}e^{-|\vec{\psi}|} \\ \quad \leftrightarrow & \frac{1}{\pi^2}\frac{d^3q}{(\vec{q}^2+1)^2} = \frac{d\Omega^3}{|\Omega^3|} \end{cases}$$

3.4.1 Representations of Cartan Spacetime

The embedding of the defining representation of the causal group, i.e. of 1-dimensional future, into Cartan future is given by the Fourier transformed advanced pole measure

$$\begin{array}{rrl} \mathbf{D}(1) \cong \mathbb{R}_\vee : & \vartheta(\tau) & = \int \frac{dq}{2i\pi}\ \frac{1}{q-io}e^{iq\tau} \\ \mathbf{D}(1)\times\mathbf{SO}_0(1,1) \cong \mathbb{R}^2_\vee : & \vartheta(x^2)\vartheta(x_0) & = -\int \frac{d^2q}{2\pi^2}\ \frac{1}{(q-io)^2}e^{iqx} \end{array}$$

Representations of the position hyperboloids $\mathcal{Y}^1$ are embedded with an energy dependent invariant for the Ω^1-measure

$$\frac{d^2q}{-q_{\rm P}^2+m_3^2} = dq_0\vartheta(Q^2)\ \frac{d\Omega^1(Q^2)}{2} + \dots, \quad d\Omega^1(Q^2) = \frac{2dq_3}{q_3^2+Q^2}, \quad Q^2 = m_3^2 - q_0^2$$

The Hilbert representations of the causal group $\mathbf{D}(1) \ni e^t \longmapsto e^{imt}$ can be reformulated in the tangent parametrization

$$\mathbb{R}_\vee \ni \vartheta(\tau)\tau \longmapsto \int \frac{dq}{2i\pi}\ \frac{q}{(q-io)^2-m^2}e^{iq\tau} = \vartheta(\tau)\cos m\tau$$

2-dimensional future $\mathbf{D}(1)\times\mathbf{D}(1) \cong \mathbf{D}(1)\times\mathbf{SO}_0(1,1)$ has rank 2. The residual representations of these two noncompact groups are characterized by two invariants (M_0^2, M_3^2) for energy and momenta, both from a continuous spectrum

$$\text{for } \mathbf{D}(1)\times\mathbf{D}(1) : \quad (t,z) \longmapsto \begin{array}{l} \int \frac{dq_0}{2i\pi}\frac{q_0}{(q_0-io)^2-M_0^2}e^{iq_0t}\int\frac{dq_3}{\pi}\frac{|M_3|}{q_3^2+M_3^2}e^{-iq_3z} \\ = \vartheta(t)\cos M_0t\ e^{-|M_3z|} \end{array}$$

In a Lorentz compatible framework the residual representations will be supported by two Lorentz invariants for the hyperbolic-spherical singularity surface with the pole function

$$\begin{array}{rl} \frac{1}{(q^2-m_0^2)(-q^2+m_3^2)} & = \frac{1}{m_0^2-m_3^2}[-\frac{1}{q^2-m_0^2}+\frac{1}{q^2-m_3^2}] \\ = -\int_{m_3^2}^{m_0^2} \frac{dm^2}{m_0^2-m_3^2} \frac{\partial}{\partial m^2}\frac{1}{q^2-m^2} & = -\int_{m_3^2}^{m_0^2} \frac{dm^2}{m_0^2-m_3^2} \frac{1}{(q^2-m^2)^2} \end{array}$$

The two invariants are incorporated in a spectral function for a dipole measure. By the Lorentz compatible embedding both invariants contribute to representations of the causal group $\mathbf{D}(1)$ and the position hyperboloid $\mathbf{SO}_0(1,1)$.

On the lightcone $x^2 = 0$, where time and position translations coincide $x^3 = \pm x^0$, the contributions from both invariants cancel each other

$$\begin{array}{rl} \text{Cartan future: } \mathbb{R}^2_\vee \ni x_\vee \longmapsto & -\frac{1}{\pi}\int \frac{d^2q}{2i\pi}\frac{(m_0^2-m_3^2)\ q\ e^{iqx}}{[(q-io)^2-m_0^2][(q-io)^2-m_3^2]} \\ = -\frac{\partial}{\partial x}[\mathcal{E}_0(\frac{m_0^2x_\vee^2}{4})-\mathcal{E}_0(\frac{m_3^2x_\vee^2}{4})] & = \frac{x_\vee}{2}[m_0^2\mathcal{E}_1(\frac{m_0^2x_\vee^2}{4})-m_3^2\mathcal{E}_1(\frac{m_3^2x_\vee^2}{4})] \end{array}$$

Representations of future have - by the corresponding tangent space integrations - projections[45] $x_\vee = \tau_\vee \mathbf{1}_2 + z\sigma_3$ on representations of the causal group $\mathbf{D}(1)$ and of the position hyperboloids $\mathcal{Y}^1$

$$\begin{array}{lrl} \text{time future:} & \mathbf{D}(1) \cong \mathbb{R}_\vee \ni \tau_\vee \longmapsto & \int \frac{dx_3}{2\pi}\int \frac{d^2q}{2i\pi}\frac{(m_0^2-m_3^2)\ q\ e^{iqx}}{[(q-io)^2-m_0^2][(q-io)^2-m_3^2]} \\ & & = \vartheta(\tau)[\cos m_0\tau - \cos m_3\tau] \\ \text{position:} & \mathcal{Y}^1 \cong \mathbb{R} \ni z \longmapsto & \int \frac{dx_0}{2\pi}\int \frac{d^2q}{2i\pi}\frac{(m_0^2-m_3^2)\ q\ e^{iqx}}{[(q-io)^2-m_0^2][(q-io)^2-m_3^2]} \\ & & = \frac{\partial}{\partial z}V(z) \end{array}$$

The position projections display exponential interactions

$$V(z) = \frac{e^{-|m_0z|}}{|m_0|} - \frac{e^{-|m_3z|}}{|m_3|},\quad \frac{\partial}{\partial z}V(z) = \epsilon(z)(e^{-|m_0z|}-e^{-|m_3z|})$$

3.4.2 Representations of Minkowski Spacetime

4-dimensional future

$$\mathbf{D}(2) \cong \mathbb{R}^4_\vee:\quad \vartheta(x^2)\vartheta(x_0) \ = \int \frac{d^4q}{2\pi^3}\frac{1}{[(q-io)^2]^2}e^{iqx}$$

is the real homogeneous space $\mathbf{D}(1)\times\mathcal{Y}^3$ with rank 2 for a Cartan subgroup $\mathbf{D}(1)\times\mathbf{SO}_0(1,1)$. Representations of the position hyperboloids $\mathcal{Y}^3$ are embedded with an energy dependent invariant

$$\frac{d^4q}{(q_{\rm P}^2-m^2)^2} = dq_0\vartheta(Q^2)\frac{d\Omega^3(Q^2)}{2}+\dots,\ d\Omega^3(Q^2) = \frac{2d^3q}{(\vec q^2+Q^2)^2} \text{ with } Q^2 = m^2-q_0^2$$

The residual representations will be supported by two invariants as for the 2-dimensional case with a characteristic additional dipole structure[10] for the 2-sphere degrees of freedom in 3-dimensional position

$$\begin{array}{rl} \text{for } \mathbf{D}(1)\times[\mathbf{D}(1)\times\Omega^2]:\quad (t,\vec x) \longmapsto & \int \frac{dq_0}{2i\pi}\frac{q_0}{(q_0-io)^2-M_0^2}e^{iq_0t}\int\frac{d^3q}{\pi^2}\frac{|M_3|}{(\vec q^2+M_3^2)^2}e^{-i\vec q\vec x} \\ & = \vartheta(t)\cos M_0t\ e^{-|M_3|r} \end{array}$$

In a Lorentz compatible formulation the two invariants can be incorporated in a spectral function for a tripole measure

$$\begin{array}{rl} \frac{1}{(q^2-m_0^2)(-q^2+m_3^2)^2} & = \frac{1}{(m_0^2-m_3^2)^2}[\frac{1}{q^2-m_0^2} - \frac{1}{q^2-m_3^2} - \frac{m_0^2-m_3^2}{(q^2-m_3^2)^2}] \\ = \int_{m_3^2}^{m_0^2} \frac{m_0^2-m^2}{(m_0^2-m_3^2)^2} dm^2 \left(\frac{\partial}{\partial m^2}\right)^2 \frac{1}{q^2-m^2} & = 2\int_{m_3^2}^{m_0^2} \frac{m_0^2-m^2}{(m_0^2-m_3^2)^2} dm^2 \ \frac{1}{(q^2-m^2)^3} \end{array}$$

The spherical invariant m_0^2 in the pole measure for $\mathbf{D}(1)$ and the hyperbolic invariant m_3^2 in the dipole measure for $\mathcal{Y}^3$ contribute also to representations of $\mathcal{Y}^3$ and $\mathbf{D}(1)$. There is one additional continuous invariant compared with the Poincaré group representations for free particles.

With two invariants the vector representations of 4-dimensional future are

$$\begin{array}{l} \text{Minkowski future: } \mathbb{R}^4_\vee \ni x_\vee \longmapsto \frac{1}{\pi}\int \frac{d^4q}{2i\pi^2} \frac{(m_0^2-m_3^2)^2 \ q \ e^{iqx}}{[(q-io)^2-m_0^2][(q-io)^2-m_3^2]^2} \\ \quad = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial \frac{x^2}{4}}[\mathcal{E}_0(\frac{m_0^2x_\vee^2}{4}) - \mathcal{E}_0(\frac{m_3^2x_\vee^2}{4})] - (m_0^2-m_3^2)\mathcal{E}_0(\frac{m_3^2x_\vee^2}{4})\right] \\ \quad = \frac{x_\vee}{2}[m_0^4\mathcal{E}_2(\frac{m_0^2x_\vee^2}{4}) - m_3^4\mathcal{E}_2(\frac{m_3^2x_\vee^2}{4}) + (m_0^2-m_3^2)m_3^2\mathcal{E}_1(\frac{m_3^2x_\vee^2}{4})] \end{array}$$

The projections $x_\vee = \tau_\vee\mathbf{1}_2 + \vec{x}$ on time future and 3-dimensional position hyperboloids

$$\begin{array}{lrl} \text{time future:} & \mathbf{D}(1) \cong \mathbb{R}_\vee \ni \tau_\vee \longmapsto & \int \frac{d^3x}{8\pi^2}\int \frac{d^4q}{2i\pi^2} \frac{(m_0^2-m_3^2)^2 \ q \ e^{iqx}}{[(q-io)^2-m_0^2][(q-io)^2-m_3^2]^2} \\ & & = \vartheta(\tau)[\cos m_0\tau - \cos m_3\tau + \frac{m_0^2-m_3^2}{2m_3^2} m_3\tau \sin m_3\tau] \\ \text{position:} & \mathcal{Y}^3 \cong \mathbb{R}^3 \ni \vec{x} \longmapsto & \int \frac{dx_0}{2\pi}\int \frac{d^4q}{2i\pi^2} \frac{(m_0^2-m_3^2)^2 \ q \ e^{iqx}}{[(q-io)^2-m_0^2][(q-io)^2-m_3^2]^2} \\ & & = (m_0^2-m_3^2)\frac{\partial}{\partial \vec{x}} V_3(r) \end{array}$$

show exponential and Yukawa interactions - the 2-sphere spread of a noncompact representation of 1-dimensional position with a z-proportional contribution from the dipole

$$\begin{array}{rl} V_3(r) & = \frac{e^{-|m_3|r}}{2|m_3|} + \frac{e^{-|m_0|r} - e^{-|m_3|r}}{(m_0^2-m_3^2)r} = -\frac{d}{dr^2}V_1(r) \\ V_1(r) & = (1+|m_3|r)\frac{e^{-|m_3|r}}{2|m_3|^3} + \frac{\frac{e^{-|m_0|r}}{|m_0|} - \frac{e^{-|m_3|r}}{|m_3|}}{m_0^2-m_3^2} \end{array}$$

3.5 Measure Convolution and Bound States

Feynman integrals involve convolutions of energy-momentum measures (distributions)

$$(\mu_1 * \mu_2)(q) = \int d^4q_1 d^4q_2 \ \mu_1(q_1)\delta(q_1+q_2-q)\mu_2(q_2)$$

For free particle energy-momentum measures

$$\delta_{\pm|m|}(q) = \vartheta(\pm q_0)\delta(q^2-m^2), \quad m \geq 0$$

the product measures display the familiar energy thresholds, e.g.

$$\begin{aligned}(\delta_{\pm|m_1|} * \delta_{\pm|m_2|})(q) &= 2\pi\vartheta(\pm q_0)\vartheta(q^2 - m_+^2)\frac{\sqrt{\Delta(q^2,m_1^2,m_2^2)}}{q^2}\\ \Delta(q^2, m_1^2, m_2^2) &= (q^2 - m_+^2)(q^2 - m_-^2), \quad m_\pm = m_1 \pm m_2\end{aligned}$$

In the convolution of two Feynman particle propagators

$$\begin{aligned}\pm\frac{1}{i\pi}\frac{1}{q^2\mp io-m_1^2} * \pm\frac{1}{i\pi}\frac{1}{q^2\mp io-m_2^2} = 2\Big[&\quad \vartheta(+q_0)\delta(q^2 - m_1^2) * \vartheta(+q_0)\delta(q^2 - m_2^2)\\ &+ \vartheta(-q_0)\delta(q^2 - m_1^2) * \vartheta(-q_0)\delta(q^2 - m_2^2)\Big]\\ &\pm \frac{1}{i\pi}\Big[\delta(q^2 - m_1^2) * \frac{1}{q_{\rm P}^2-m_2^2} + \frac{1}{q_{\rm P}^2-m_1^2} * \delta(q^2 - m_2^2)\Big]\end{aligned}$$

the on shell-off shell convolution of the Dirac particle measures with principal value interaction measures is not defined (‘divergencies’). That is different for the higher order interaction measures above.

Product representations can be related to measure products as seen in the simplest example

$$\begin{aligned}e^{iE_1t}e^{iE_2t} &= e^{iE_+t}, \quad (\delta_{E_1} * \delta_{E_2})(q_0) = \delta_{E_+}(q_0)\\ e^{iEt} &= \int dq_0\delta(q_0 - E)e^{iq_0t}, \quad E_+ = E_1 + E_2\end{aligned}$$

Therewith bound state structures for particles are proposed to be derived from pole invariants in convolution products involving the interaction measures as described above. First steps on this way have been tried elsewhere[45].

Chapter 4

Intertwining Spacetime and Hyperisospin

If spacetime is modeled by the equivalence classes $\mathbf{GL}(\mathbb{C}^2)/\mathbf{U}(2)$, the class characterizing fixgroup $\mathbf{U}(2)$ arises with local internal operations for the spacetime fields.

The square integrable $\mathbb{C}$-valued functions $L^2(G/U)$ on a symmetric space can be rearranged with respect to mappings of the fixgroup classes G/U into Hilbert spaces W with nontrivial representations of the fixgroup $U \subseteq G$. A familiar example is given with the defining representation of the rotation group

$$\left(\begin{array}{c|c|c} e^{i(\chi+\varphi)}\cos^2\frac{\theta}{2} & ie^{i\varphi}\frac{\sin\theta}{\sqrt{2}} & -e^{-i(\chi-\varphi)}\sin^2\frac{\theta}{2} \\ ie^{i\chi}\frac{\sin\theta}{\sqrt{2}} & \cos\theta & ie^{-i\chi}\frac{\sin\theta}{\sqrt{2}} \\ -e^{i(\chi-\varphi)}\sin^2\frac{\theta}{2} & ie^{-i\varphi}\frac{\sin\theta}{\sqrt{2}} & e^{-i(\chi+\varphi)}\cos^2\frac{\theta}{2} \end{array}\right) \in \mathbf{SO}(3)$$

The three columns as vectors of 3-dimensional spaces are acted on with the defining $\mathbf{SO}(3)$-representation. The middle column is the spherical harmonic $\sqrt{\frac{4\pi}{3}}\mathrm{Y}^1(\frac{\vec{x}}{r}) \in W_0 \cong \mathbb{C}^3$ which, by its products, builds up $L^2(\mathbf{SO}(3)/\mathbf{SO}(2))$ which carries a trivial representation of the fixgroup $e^{i\chi} \in \mathbf{SO}(2)$. This has been considered in the former chapter. The left and right column with $\chi = 0$ describe mappings $w : \Omega^2 \longrightarrow W_\pm$ into vector spaces $W_\pm \cong \mathbb{C}^3$ acted upon with the nontrivial $\mathbf{SO}(2)$-representations $\{e^{\pm i\chi}\}$. Other examples with particle fields are given below.

The general mathematical procedure behind all this is the inducing of G-representations from subgroup U-representations [28, 15, 3]. In nonlinear spacetime as the orientation manifold $\mathbf{D}(2) \cong \mathbf{GL}(\mathbb{C}^2)/\mathbf{U}(2)$, the local compact fixgroup $\mathbf{U}(2)$ will be interpreted as hyperisospin. A harmonic analysis of spacetime interactions employs only such representations of the extended Lorentz group $\mathbf{GL}(\mathbb{C}^2)$ which are induced from representations of the hyperisospin group $\mathbf{U}(2)$.

4.1 Dichotomy of Spacetime and Chargelike Operations

The real 8-dimensional linear group in two complex dimensions is, via the polar decomposition

$$\mathbf{GL}(\mathbb{C}^2) = \mathbf{U}(2) \circ \mathbf{D}(2) \ni g = u(g) \circ |g|, \quad |g| = x_\vee = \vartheta(x^2)\vartheta(x_0)x \in \mathbf{D}(2)$$

the product of its maximal compact group $\mathbf{U}(2)$ (hyperisospin), comprising the four internal or chargelike degrees of freedom, with all $\mathbf{U}(2)$-orbits, the noncompact causal symmetric space $\mathbf{D}(2)$ (nonlinear spacetime), comprising the four external or spacetime related degrees of freedom. Elementary interactions and elementary particles are characterized by different symmetries: Elementary interactions display a dichotomic nonabelian symmetry with internal (chargelike) and external (spacetime-like) operations

$$\text{interaction symmetry: } \mathbf{U}(2) \times \mathbf{GL}(\mathbb{C}^2)$$

exemplified by the subgroup product $\mathbf{SU}(2)\times\mathbf{SU}(2)$ with isospin and spin. The action group is a direct product of the hyperisospin group and the extended Lorentz group

$$\begin{array}{lrll} \text{internal} & \mathbf{U}(2) & = \mathbf{U}(\mathbf{1}_2) \circ \mathbf{SU}(2) & \cong \frac{\mathbf{U}(1)\times\mathbf{SU}(2)}{\mathbf{I}(2)} \\ \text{external} & \mathbf{GL}(\mathbb{C}^2) & = \mathbf{D}(1) \times \mathbf{U}(\mathbf{1}_2) \circ \mathbf{SL}(\mathbb{C}^2) & \cong \mathbf{D}(1) \times \frac{\mathbf{U}(1)\times\mathbf{SL}(\mathbb{C}^2)}{\mathbf{I}(2)} \end{array}$$

$\mathbb{I}(2) = \{\pm 1\}$ denotes the discrete center of the special groups, i.e. abelian and nonabelian transformations are centrally correlated

$$\mathbf{U}(\mathbf{1}_2) \cap \mathbf{SU}(2) \cong \mathbb{I}(2) \cong \mathbf{U}(\mathbf{1}_2) \cap \mathbf{SL}(\mathbb{C}^2)$$

In such an interpretation, the observed interaction symmetries, e.g. isospin and spin $\mathbf{SU}(2) \times \mathbf{SU}(2)$, originate from one group represented in two different actions as internal and external group. The dichotomic interaction symmetry disappears in the symmetry for free particles. In the standard model formulation, the isospin $\mathbf{SU}(2)$-symmetry is 'broken' (bleached), the spin $\mathbf{SU}(2)$-symmetry survives[42].

4.2 Induced Representations

The doubling of a group G to a dichotomic operation group $G \times G$ arises as a basic general mathematical structure: The binary property of the product in a group $G \times G \longrightarrow G$ allows the realization of the group on itself by both left and right multiplications, i.e. by the realization of the square group

$$G \times G \ni (u,k) \longmapsto L_u \times R_k \text{ with } L_u \circ R_k(g) = ugk^{-1} \text{ for } g \in G$$

and subgroups $U \times G \subseteq G \times G$. E.g., the diagonal subgroup $(k,k) \in (G \times G)_\Delta \cong G$ is realized by the inner group automorphisms $\mathrm{Int}\, k = L_k \circ R_k$. Or, a

matrix can be acted upon from left and right - with possibly different groups $U \times G$. This binary structure and the related two-sided regular representation, of special interest for nonabelian operations, $\mathrm{Int}\, k(g) = kgk^{-1} \neq g$, is used in the method of G-representations induced from subgroup U-representations.

The inducing procedure uses a factorization[1] $G = U \times U \setminus G$ with subgroup U-classes. In the spacetime application the extended Lorentz group $G = \mathbf{GL}(\mathbb{C}^2)$ is factorized into $\mathbf{U}(2) \times \mathbf{D}(2)$ with the hyperisospin subgroup $U = \mathbf{U}(2)$ and the future cone $\mathbf{D}(2)$. On the factorization $U \times U \setminus G$ one constructs representations of the subgroup $U \times G \subseteq G \times G$, here representation of the internal-external operation group of the interactions.

A group representation on $U \times U \setminus G$ involves subgroup U-intertwiners. The U-intertwiners $\mathbf{B}$ of a group G and a vector space W with U-representation $D : U \longrightarrow \mathbf{GL}(W)$ are group orbits $G \longrightarrow W$, compatible with the U-action (taken as left action)

$$\begin{array}{ccc} G & \xrightarrow{L_u} & G \\ \mathbf{B} \downarrow & & \downarrow \mathbf{B} \\ W & \xrightarrow[D(u)]{} & W \end{array} , \quad \begin{array}{l} u \in U, \ g \in G \\ \mathbf{B}(g) = D(u)\mathbf{B}(u^{-1}g) \end{array}$$

The intertwiners constitute the U-invariant subspace $W^{U \setminus G}$ in the vector space W^G with all mappings. The dimension of $W^{U \setminus G}$ is the cardinality of the symmetric space $U \setminus G$ times the dimension of the U-representation space W. The intertwiners can be looked at to be mappings on the classes $U \setminus G$, i.e. from the subgroup U-orbits in the group G into the U-orbits in the vector space W

$$U \setminus G \longrightarrow V/U, \quad Ug \longmapsto \mathbf{B}(Ug) = D[U]\mathbf{B}(g)$$

With a G-invariant measure $d\mu(Ug)$ of the U-orbits the intertwiners can be expanded as direct integral with the values on the classes

$$\mathbf{B} = {}^{\oplus}\!\!\int_{U \setminus G} d\mu(Ug)\ \mathbf{B}(Ug) = {}^{\oplus}\!\!\int_{U \setminus G} d\mu(Ug)\ \mathrm{e}(Ug)^a \mathbf{B}(Ug)_a$$

Here $\{\mathrm{e}^a\}$ is a basis of W, $\{\mathrm{e}(Ug)^a\}$ a basis distribution of the representation space at each class $W(Ug) = W \times \{Ug\} \cong W$ and $\mathbf{B}(Ug)_a$ the corresponding coefficients.

The G-representation, induced from the U-representation on W, is the right action with the full group G on the U-intertwiners

$$k \in G: \quad W^{U \setminus G} \longrightarrow W^{U \setminus G}, \quad \mathbf{B} \longmapsto \mathbf{B}_k = {}^{\oplus}\!\!\int_{U \setminus G} d\mu(Ug)\ \mathrm{e}(Ug)^a \mathbf{B}(Ugk)_a$$

The right G-action $U \setminus G \times G \longrightarrow U \setminus G$ has U-isomorphic fixgroups.

In general, the induced G-representation on the intertwiner vector space $W^{U \setminus G}$ is highly decomposable

$$\begin{aligned} W^{U \setminus G} &= \bigoplus_I W^I, \quad I \in \mathbf{rep}\, G \\ \mathbf{B} &= \bigoplus_I {}^{\oplus}\!\!\int_{U \setminus G} d\mu(Ug)\ \mathbf{B}(Ug)^I = \bigoplus_I {}^{\oplus}\!\!\int_{U \setminus G} d\mu(Ug)\ \mathrm{e}(Ug)^a \mathbf{B}(Ug)^I_a \end{aligned}$$

[1] Obviously, a factorization with right or with left cosets is equivalent, $U \times U \setminus G \cong G \cong G/U \times U$.

For compact groups $G \supseteq U$ one has Frobenius' theorem[4, 3] for the multiplicity $n(D_G)$ of an irreducible G-representation D_G, induced by an irreducible U-representation D_U - given by $n(D_G) = n(D_G/D_U)$, i.e. the multiplicity of D_U in D_G.

The expansion coefficients $\mathbf{B}(Ug)^I_a$, called transmutators [42, 43], constitute - for finite dimensional G and U-representations - rectangular matrices, e.g. the (1×3)-matrices in the chapter introduction. They have a typical hybrid left-right transformation behavior for the product group $U \times G$.

4.2.1 Free Particle Fields as Spin Intertwiners

Familiar examples for intertwiners and the inducing procedure are the particle fields. Particle fields use the inducing procedure for the Poincaré group via the Lorentz group action on the energy-momenta with the fixgroups ('little groups') $\{\mathbf{SO}_0(1,3), \mathbf{SO}_0(1,2), \mathbf{SO}(3), \mathbf{SO}(2)\ \vec{\times}\, \mathbb{R}^2\}$.

The representations of the Poincare group $G = \mathbf{SL}(\mathbb{C}^2)\ \vec{\times}\, \mathbb{R}^4$ are induced, for massive particles, from $U = \mathbf{SU}(2) \times \mathbb{R}^4$-representations. The inducing procedure involves the spin classes in the Lorentz group

$$\mathbf{SL}(\mathbb{C}^2) = \mathbf{SL}(\mathbb{C}^2)/\mathbf{SU}(2) \times \mathbf{SU}(2)$$

Particle fields are spin intertwiners and can be written as a direct integral with Lorentz invariant boost measure $\frac{d^3q}{q_0}$, parametrized by the momenta $\frac{\vec{q}}{m} \in \mathbf{SL}(\mathbb{C}^2)/\mathbf{SU}(2) \cong \mathbb{R}^3$ of the mass hyperboloid $q^2 = m^2$. Creation and annihilation operators give a basis distribution for the spin J representation spaces $W(\vec{q}) \cong \mathbb{C}^{1+2J} \times \{\vec{q}\}$ for each momentum, e.g. the electron-positron

$$\mathbf{\Psi} \;=\begin{pmatrix} \mathbf{l}^{\dot{A}} \\ \mathbf{r}^{A} \end{pmatrix} = \sqrt{2m}\ {}^{\oplus}\!\!\int_{\mathbb{R}^3} \frac{d^3q}{2q_0(2\pi)^3} \begin{pmatrix} s(\frac{q}{m})^{\dot{A}}_a \frac{\mathrm{u}(\vec{q})^a + \mathrm{a}^\star(\vec{q})^a}{\sqrt{2}} \\ \hat{s}(\frac{q}{m})^{A}_a \frac{\mathrm{u}(\vec{q})^a - \mathrm{a}^\star(\vec{q})^a}{\sqrt{2}} \end{pmatrix} \text{ with } q_0 = \sqrt{m^2 + \vec{q}^2}$$

The expansion coefficients (spin-Lorentz group transmutators) are the fundamental left and right handed (2×2)-Weyl boost representations

$$s(\tfrac{q}{m})^{\dot{A}}_a \cong \sqrt{\tfrac{m+q_0}{2m}}(\mathbf{1}_2 + \tfrac{\vec{q}}{m+q_0}) \text{ with } \left\{ \begin{array}{rrl} \text{spin } \mathbf{SU}(2): & a & = 1,2 \\ \text{Lorentz } \mathbf{SL}(\mathbb{C}^2): & \dot{A} & = 1,2 \end{array} \right.$$

The indices are of 'different quality' - for a vector space $V \cong \mathbb{C}^2$ with Lorentz group Weyl representation and for the inducing vector space $W \cong \mathbb{C}^2$ with spin $J = \frac{1}{2}$ representation.

With the action of the product group $\mathbf{SL}(\mathbb{C}^2) \times \mathbf{SU}(2)$ there are two different transformation structures: The transmutators, parametrized with momenta, are acted upon with the extended Lorentz group (from left). This produces the boosts for the momenta in $q \longmapsto \lambda q \lambda^\star$ and the Wigner $\mathbf{SU}(2)$-rotation (from right), dependent on the acting λ and the class $\frac{\vec{q}}{m}$

$$\lambda \in \mathbf{SL}(\mathbb{C}^2): \quad \lambda \circ s(\tfrac{q}{m}) = s(\lambda \tfrac{q}{m} \lambda^\star) \circ r(\lambda, \tfrac{q}{m}), \quad r(\lambda, \tfrac{q}{m}) \in \mathbf{SU}(2)$$

Under Lorentz group action the creation-annihilation operators transform with the corresponding Wigner $\mathbf{SU}(2)$-rotation

$$\begin{array}{rl} \lambda \in \mathbf{SL}(\mathbb{C}^2): \quad \mathrm{u}(\lambda\vec{q}\lambda^*)^a & = r(\lambda, \frac{q}{m})^a_b \mathrm{u}(\vec{q})^b \\ \Rightarrow \mathbf{l}^{\dot{A}} & \longmapsto \lambda^{\dot{A}}_{\dot{B}} \mathbf{l}^{\dot{B}} \end{array}$$

Products of the fundamental boost representations give all $\mathbf{SL}(\mathbb{C}^2)/\mathbf{SU}(2)$ transmutators, e.g. the rectangular (1×4) and (3×4) representations in $\mathbf{SO}_0(1,3)/\mathbf{SO}(3)$ by the 1st and the last three columns of the (4×4)-matrix

$$\Lambda(\tfrac{q}{m})^j_k \cong \tfrac{1}{2}\,\mathrm{tr}\, s(\tfrac{q}{m})\sigma^j s^\star(\tfrac{q}{m})\check{\sigma}_k,\quad \Lambda(\tfrac{q}{m}) = \left(\begin{array}{c|c} \frac{q_0}{m} & \frac{q_a}{m} \\ \frac{q_b}{m} & \delta_{ab} + \frac{q_a q_b}{m(m+q_0)} \end{array}\right)$$

which are used to induce Poincaré group representations for particles with spin $J=0$ (massive scalar particles) and $J=1$ (massive vector particles) resp. This is in analogy to the column decomposition of the $\mathbf{SO}(3)$-example in the chapter introduction.

4.2.2 Interaction Fields as Hyperisospin Interwiners

Interaction fields use the inducing procedure for the extended Lorentz group action arising from hyperisospin representations.

Hyperisospin interwiners $\mathbf{B}$ relate nonlinear spacetime $x_\vee \in \mathbf{D}(2) \cong \mathbf{U}(2)\backslash \mathbf{GL}(\mathbb{C}^2)$, parametrizing the hyperisospin orbits $\mathbf{U}(2)g$ in the polar decomposition $\mathbf{GL}(\mathbb{C}^2) = \mathbf{U}(2)\times\mathbf{D}(2)$, to hyperisospin $\mathbf{U}(2)$-orbits in a representation space $W\cong\mathbb{C}^n$. They can be expanded with a direct integral over the future cone

$$\mathbf{B} = {}^\oplus\!\!\int_{\mathbb{R}^4_\vee} \frac{d^4x}{(x^2)^2}\ \mathrm{e}(x)^\alpha \mathbf{B}(x)_\alpha$$

with invariant measure of nonlinear spacetime

$$\mathbb{R}^4_\vee \ni x = e^{\psi_0+\vec{\psi}}:\ \frac{d^4x}{(x^2)^2} = d\psi_0\ \sinh^2\psi\ d\psi\ d\Omega^2 = d\psi_0\ \psi^2 d\psi\ d\Omega^2 + \dots$$

$\{\mathrm{e}(x)^\alpha\}$ is a basis of the space $W(x)$ with $\mathbf{U}(2)$-representation at each point of the forward cone.

The induced representation on the hyperisospin intertwiner vector space

$$\mathbf{B}\in W^{\mathbb{R}^4_\vee} = {}^\oplus\!\!\int_{\mathbb{R}^4_\vee} \frac{d^4x}{(x^2)^2}\ W(x),\quad W(x)\cong W$$

is decomposable. A decomposition into nondecomposable $\mathbf{GL}(\mathbb{C}^2)$-representation spaces contains only such Lorentz group $\mathbf{SL}(\mathbb{C}^2)$-representations which have the inducing isospin $\mathbf{SU}(2)$-representation as spin $\mathbf{SU}(2)$-subrepresentation. E.g. an internal $\mathbf{U}(2)$-representation with isopin doublets (triplets) induces only external $\mathbf{GL}(\mathbb{C}^2)$-representations involving spin doublets (triplets).

This representation inclusion property relating external and internal interaction symmetries can be checked: Indeed, with the exception of the Higgs field, all interaction parametrizing fields of the minimal standard model have this property. The isospin $\mathbf{SU}(2)$-representations are denoted by their invariant T

$$[2T]:\mathbf{SU}(2)\longrightarrow\mathbf{SU}(1+2T)\text{ with } T\in\{0,\tfrac{1}{2},\dots\}$$

and the Lorentz group $\mathbf{SL}(\mathbb{C}^2)$-representations by their left and right-handed invariants J_L, J_R

$$[2J_L|2J_R]:\mathbf{SL}(\mathbb{C}^2)\longrightarrow\mathbf{SL}(\mathbb{C}^{(1+2J_L)(1+2J_R)})\text{ with } J_L,J_R\in\{0,\tfrac{1}{2},\dots\}$$

The Lorentz group representations are spin $\mathbf{SU}(2)$-decomposable

$$[2J_L|2J_R] \overset{\mathbf{SU}(2)}{\cong} \bigoplus_{J=|J_L-J_R|}^{J_L+J_R} [2J]$$

The standard model fields, left and right handed fermions and gauge fields, are acted upon with the following isospin-spin representations

interaction field	isospin $\mathbf{SU}(2),\ [2T]$	Lorentz group $\mathbf{SL}(\mathbb{C}^2),\ [2J_L\|2J_R]$	hypercharge $\mathbf{U}(1),\ Y$	color $\mathbf{SU}(3),\ [2C_1, 2C_2]$
left leptons $\mathbf{l}(x)$	[1]	[1|0]	$-\frac{1}{2}$	$[0,0]$
right lepton $\mathbf{r}(x)$	[0]	[0|1]	-1	$[0,0]$
left quarks $\mathbf{q}(x)$	[1]	[1|0]	$\frac{1}{6}$	$[1,0]$
right quarks $\mathbf{u}(x), \mathbf{d}(x)$	[0]	[0|1]	$\frac{2}{3}, -\frac{1}{3}$	$[1,0]$
hypercharge gauge $\mathbf{A}_0(x)$	[0]	[1|1]	0	$[0,0]$
isospin gauge $\vec{\mathbf{A}}(x)$	[2]	[1|1]	0	$[0,0]$
color gauge $\mathbf{g}(x)$	[0]	[1|1]	0	$[1,1]$
Higgs $h(x)$	[1]	[0|0]	$\frac{1}{2}$	$[0,0]$

isospin induced spin representations
$\mathbf{SU}(2)$ induces $\mathbf{SL}(\mathbb{C}^2)$ with $T \le J_{\max} = J_L + J_R$

With the exception of the Higgs field, all isospin representations for the interaction fields are Lorentz group subrepresentations. With $\mathbf{SU}(3)$ no subgroup of $\mathbf{GL}(\mathbb{C}^2)$, the color degrees of freedom are not involved in the Lorentz group representation inducing procedure. The special role of the Higgs degrees of freedom will be discussed in the next section.

4.3 The Higgs Hilbert Space

Inducing a G-representation from a subgroup U-representation on a Hilbert space $W \cong \mathbb{C}^n$

$$\begin{array}{l} W \times W \longrightarrow \mathbb{C},\ \ \langle \mathbf{B}|\mathbf{B}'\rangle \text{ with } \langle \mathbf{B}|\mathbf{B}\rangle > 0 \text{ and } \langle \mathbf{B}|\mathbf{B}\rangle = 0 \iff \mathbf{B} = 0 \\ \text{Hilbert basis: } \langle \mathrm{e}^a|\mathrm{e}^b\rangle = \delta^{ab},\ \ \mathbf{B} = \mathrm{e}^a\mathbf{B}_a,\ \ \langle \mathbf{B}|\mathbf{B}'\rangle = \overline{\mathbf{B}_a}\mathbf{B}'_a \end{array}$$

the intertwiners inherit, with a positive invariant $U \setminus G$ measure $d\mu(Uk)$, a scalar product

$$\begin{array}{rl} \mathbf{B} & = {}^{\oplus}\!\int_{U\setminus G} d\mu(Ug)\ \mathrm{e}(x)^\alpha \mathbf{B}(Ug)_a \\ W^{U\setminus G} \times W^{U\setminus G} \longrightarrow \mathbb{C},\ \langle \mathbf{B}|\mathbf{B}\rangle & = \int_{U\setminus G} d\mu(Ug)\ \overline{\mathbf{B}(Ug)_a}\mathbf{B}'(Ug)_a \end{array}$$

In contrast to compact groups, the scalar product for a noncompact group G has not to be finite if restricted to G-subrepresentations $W^I \subset W^{U\setminus G}$. E.g., the Weyl boost representations $s(\frac{q}{m})$ are not $\frac{d^3q}{q_0}$-square integrable - momentum wave packets have to be used.

The basis distributions $\{\mathrm{e}(Ug)^a\}$ have 'continuous orthonormality'

$$\langle \mathrm{e}(Ug)^a|\mathrm{e}(Ug')^b\rangle = \delta^{ab}\delta(\mu(Ug, Ug'))$$

with a Dirac distribution, associated to the $U \setminus G$-measure

$$\int_{U\setminus G \times U\setminus G} d\mu(Ug)d\mu(Ug')\ f(Ug, Ug')\delta(\mu(Ug, Ug')) = \int_{U\setminus G} d\mu(Ug)\ f(Ug, Ug)$$

The particle fields above are examples with the spin Hilbert spaces $W \cong \mathbb{C}^{1+2J}$, e.g. the electron-positron field

$$\begin{array}{rl} \mathbf{l}^{\dot{A}} & = \sqrt{2m} \; {}^{\oplus}\!\!\int_{\mathbf{R}^3} \frac{d^3q}{2q_0(2\pi)^3} s(\frac{q}{m})^{\dot{A}}_a \frac{\mathrm{u}(\vec{q})^a + \mathrm{a}^\star(\vec{q})^a}{\sqrt{2}} \text{ with } q_0 = \sqrt{m^2 + \vec{q}^2} \\ \mathbf{SL}(\mathbb{C}^2)/\mathbf{SU}(2) & \cong \mathcal{Y}^3 \cong \mathbb{R}^3 \ni \frac{\vec{q}}{m} \longmapsto \mathrm{u}(\vec{q})^a \in W \cong \mathbb{C}^2 \\ & \langle \mathrm{u}(\vec{q})^a | \mathrm{u}(\vec{q}\,')^b \rangle = \delta^{ab} 2q_0 \delta(\frac{\vec{q}-\vec{q}\,'}{2\pi}) \end{array}$$

What the momentum dependent creation-annihilation operators are for the induced Poincaré group representation, that is the spacetime dependent Higgs field which induces extended Lorentz group $\mathbf{GL}(\mathbb{C}^2)$-representations from hyperisospin $\mathbf{U}(2)$-representations. The Higgs field is a basis distribution for the Hilbert space $H \cong \mathbb{C}^2$ with the defining $\mathbf{U}(2)$-representation

$$\begin{array}{rl} \mathbf{B} & = {}^{\oplus}\!\!\int_{\mathbf{R}^4_\vee} \frac{d^4x}{(x^2)^2} \;\; h(x)^\alpha \mathbf{B}(x)_\alpha \\ \mathbf{U}(2) \backslash \mathbf{GL}(\mathbb{C}^2) & \cong \mathbb{R}^4_\vee \ni x \longmapsto h(x)^\alpha \in H \cong \mathbb{C}^2 \\ & \langle h(x)^\alpha | h(x')^\beta \rangle = \delta^{\alpha\beta} (x^2)^2 \vartheta(z^2) \delta(z), \quad z = x - x' \end{array}$$

E.g., the left lepton fields can arise as an irreducible $\mathbf{GL}(\mathbb{C}^2)$-representation induced by the Higgs field

$$\mathbf{l}^{\dot{A}} = {}^{\oplus}\!\!\int_{\mathbf{R}^4_\vee} \frac{d^4x}{(x^2)^2} \; h(x)^\alpha \mathbf{l}(x)^{\dot{A}}_\alpha \text{ with } \left\{ \begin{array}{rl} \text{Lorentz } \mathbf{SL}(\mathbb{C}^2): & \dot{A} = 1,2 \\ \text{isospin } \mathbf{SU}(2): & \alpha = 1,2 \end{array} \right.$$

$\mathbf{l}^{\dot{A}}$ has two expansions: A Higgs expansion for nonlinear spacetime $x \in \mathbb{R}^4_\vee$ as interaction field with hyperisospin action, and a creation-annihilation expansion for its momentum hyperboloid $\frac{\vec{q}}{m} \in \mathcal{Y}^3$ as particle field with Poincaré group action.

4.4 A Fermi-Clifford Algebra for Hyperisospin

In the last section an attempt will be described to induce all standard model interaction fields from $\mathbf{U}(2)$-representations on the Higgs degrees of freedom.

The hypercharges of the interaction fields in the standard model

$$Y \in \{0, \pm\tfrac{1}{6}, \pm\tfrac{1}{3}, \pm\tfrac{1}{2}, \pm\tfrac{2}{3}, \pm 1\}$$

suggest the grading structure of a Grassmann algebra whose basic vector space is complex 2-dimensional and will be identified with the Higgs Hilbert space $H \cong \mathbb{C}^2$ and its dual $H^\star \cong \mathbb{C}^2$

$$\mathbf{U}(2) \times (H \;\oplus\; H^\star) \longrightarrow H \;\oplus\; H^\star, \quad \left\{ \begin{array}{rl} u.H & = e^{i\alpha_0 + i\vec{\alpha}} H \\ u.H^\star & = H^\star e^{-i\alpha_0 - i\vec{\alpha}} \end{array} \right.$$

$$\text{hypercharge-isospin invariants: } [2Y; 2T] = \left\{ \begin{array}{rl} [1;1] & \text{for } H \\ [-1;1] & \text{for } H^\star \end{array} \right.$$

bases for H, $H^\star$: $\{h^\alpha, h^\star_\alpha \,|\, \alpha = 1,2\}$

The Grassmann algebra $\bigwedge[H \ \oplus \ H^\star]$ of Higgs and anti-Higgs Hilbert space is the direct sum of their totally antisymmetrized tensor products. It has complex dimension 2^4 and comes with a $\mathbb{Z}_5$-graduation, given by the hypercharge invariants $2Y \in \mathbb{Z}_5 = \{0, \pm 1, \pm 2\}$. The $\binom{4}{2-2|Y|}$-dimensional spaces with grade $2Y$ are decomposable into irreducible isospin representation spaces with dimension $1+2T$

	$2T = 0$	$2T = 1$	$2T = 2$
$2Y = +2$	$h^\alpha \epsilon_{\alpha\beta} h^\beta$	$-$	$-$
$2Y = -2$	$h^\star_\alpha \epsilon^{\alpha\beta} h^\star_\beta$	$-$	$-$
$2Y = +1$	$-$	$h^\alpha, \ \ h^\star_\alpha h^\gamma \epsilon_{\gamma\beta} h^\beta$	$-$
$2Y = -1$	$-$	$h^\star_\alpha, \ \ h^\alpha h^\star_\gamma \epsilon^{\gamma\beta} h^\star_\beta$	$-$
$2Y = 0$	$1, \ h^\star_\alpha h^\alpha, \ (h^\star_\alpha h^\alpha)^2$	$-$	$h^\star_\alpha \vec{\tau}^\alpha_\beta h^\beta$

hyperisospin properties of the Higgs-Grassmann algebra
$\bigwedge(H \ \oplus \ H^\star) \cong \mathbb{C}^{16}$

The Grassmann vector space is endowed with a Fermi structure[30] by nontrivial anticommutators for the generating Higgs vectors

$$\text{Fermi anticommutators:} \quad \{h^\star_\alpha, h^\beta\} = \delta^\beta_\alpha, \quad \{h^\alpha, h^\beta\} = 0, \quad \{h^\star_\alpha, h^\star_\beta\} = 0$$

Bose and Fermi elements have even and odd grades resp. in the Fermi quantum algebra $\mathbf{Q}_+(\mathbb{C}^2) \cong \mathbb{C}^{16}$ which is the vector space structure of the Grassmann algebra with nontrivial basic anticommutators[33].

The Fermi quantum algebra is isomorphic to a complexified Clifford algebra with neutral signature $\mathbf{O}(2,2)$

$$\bigwedge(H \ \oplus \ H^\star) \cong \mathbf{Q}_+(\mathbb{C}^2) \cong \mathbb{C} \otimes \mathrm{CLIFF}(2,2), \quad \mathrm{CLIFF}(2,2) \cong \mathbb{R}^{16}$$

This can be seen with a basis, (anti)-hermitian with respect to the indefinite unitary conjugation, compatible with isopin $\mathbf{SU}(2)$

$$h^\beta \overset{\star}{\leftrightarrow} \epsilon^{\beta\delta} h^\star_\delta : \quad \begin{cases} \{\frac{h^\star_\alpha + h^\gamma \epsilon_{\gamma\alpha}}{\sqrt{2}}, \frac{h^\beta + \epsilon^{\beta\delta} h^\star_\delta}{\sqrt{2}}\} & = \delta^\beta_\alpha \\ \{\frac{h^\star_\alpha - h^\gamma \epsilon_{\gamma\alpha}}{i\sqrt{2}}, \frac{h^\beta - \epsilon^{\beta\delta} h^\star_\delta}{i\sqrt{2}}\} & = -\delta^\beta_\alpha \\ \{\frac{h^\star_\alpha - h^\gamma \epsilon_{\gamma\alpha}}{i\sqrt{2}}, \frac{h^\beta + \epsilon^{\beta\delta} h^\star_\delta}{\sqrt{2}}\} & = 0 \end{cases}$$

$\mathrm{CLIFF}(2,2)$ is isomorphic to the real (4×4)-matrices[14, 17, 44].

The $\mathbb{Z}_5$-graduation with the power difference of Higgs and anti-Higgs space reflects the adjoint action of the hypercharge $\mathbf{U}(1)$ generator

$$I = 2Y = \frac{[h^\star_\alpha, h^\alpha]}{2} \Rightarrow [I, h^\alpha] = h^\alpha, \quad [I, h^\star_\alpha] = -h^\star_\alpha$$

Isospin $\mathbf{SU}(2)$ has the generators

$$\vec{T} = \frac{h^\star_\alpha \vec{\tau}^\alpha_\beta h^\beta}{2} \Rightarrow [\vec{T}, h^\alpha] = \frac{\vec{\tau}^\alpha_\beta}{2} h^\beta, \quad [\vec{T}, h^\star_\beta] = -\frac{\vec{\tau}^\alpha_\beta}{2} h^\star_\alpha$$

with the Casimir value given by the doubled adjoint action $[\vec{T}, [\vec{T}, a]] = T(1+T)a$.

The hyperisopin representations can be related to and may induce the extended Lorentz group $\mathbf{GL}(\mathbb{C}^2)$-representations[39] by expanding the standard model interaction fields with Higgs vector products as follows

$\mathbf{U}(2)$	$\mathbf{GL}(\mathbb{C}^2)$
h^α	$\mathbf{l}_\alpha^{\dot{A}}$
$h^\star_\alpha h^\alpha$	$\mathbf{A}_k^0$
$h^\star_\alpha h^\beta \vec{\tau}^\alpha_\beta$	$\vec{\mathbf{A}}_k$
$h^\star_\beta h^\gamma h^\delta \epsilon^{\alpha\beta}\epsilon_{\gamma\delta}$	$\mathbf{Q}_\alpha^{\dot{A}}$
$h^\star_\alpha h^\star_\beta h^\gamma h^\delta \epsilon^{\alpha\beta}\epsilon_{\gamma\delta}$	$\mathbf{G}$

spacetime fields induced by the Higgs-Grassmann algebra

e.g. $^\oplus\!\int_{\mathbf{R}^4_\vee} \frac{d^4x}{(x^2)^2}\ (h^\star hh)(x)^\alpha \mathbf{Q}(x)^{\dot{A}}_\alpha$

The standard model interaction fields are transmutators from the hyperisospin $\mathbf{U}(2)$ group to the extended Lorentz group $\mathbf{GL}(\mathbb{C}^2)$. The Higgs field products involving more than one Higgs h or more than one anti-Higgs $h^\star$ induce interaction fields which can come with an additional degree of freedom. E.g., the field $\mathbf{Q}$, induced by a three Higgs's product, allows for its cubic root $\mathbf{q}$ an internal degree of freedom whereof the product has to be a singlet, e.g., it is a left handed quark isodoublet $\mathbf{q}$ with hypercharge color $\mathbf{U}(3)$ as cubic root of $\mathbf{U}(1)$

$$\begin{array}{rlcrl} \mathbf{q} & = [\mathbf{Q}]^{\frac{1}{3}}, & \text{with} & \mathbf{Q} & = \mathbf{q}\wedge\mathbf{q}\wedge\mathbf{q} \\ \mathbf{U}(3) & = [\mathbf{U}(1)]^{\frac{1}{3}} & \text{with} & \mathbf{U}(1) & = \mathbf{U}(3)\wedge\mathbf{U}(3)\wedge\mathbf{U}(3) \end{array}$$
$$^\oplus\!\int_{\mathbf{R}^4_\vee} \frac{d^4x}{(x^2)^2}\ (h^\star hh)(x)^\alpha \mathbf{Q}(x)^{\dot{A}}_\alpha = {}^\oplus\!\int_{\mathbf{R}^4_\vee} \frac{d^4x}{(x^2)^2}\ (h^\star hh)(x)^\alpha (\mathbf{q}\wedge\mathbf{q}\wedge\mathbf{q})^{\dot{A}}_\alpha$$

A cubic root of a group can be defined via functions on the spectrum. The new degrees of freedom arise in analogy to the complex phases in the cyclotomic groups $1^{\frac{1}{n}} = \{e^{2\pi i \frac{k}{n}} \mid k = 1,\dots,n\}$. Analogously, the square root $\mathbf{g} = \sqrt{\mathbf{G}}$ of a four Higgs's product allows the Lorentz vector and color octet representation properties for a gluon field $\mathbf{g}(x)^a_k$ with $\mathbf{G} = \mathbf{g}^a_k \eta^{kl}\delta_{ab}\mathbf{g}^b_l$. In such an approach, color nontrivial fields for an effective approximation of the interaction would be cubic and square roots of fields without color. The hypercharge properties of the color nontrivial fields match with such root properties[22, 32, 34, 35, 38].

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